Appendix A

Mathematical Concepts

Some of the mathematical concepts used in this book are summarized in this appendix. Section A.1 includes some formulas for geometry that are generally useful in machine vision, Section A.2 covers linear spaces, and Section A.3 describes the variational calculus which is used in solving ill-posed problems through regularization.

A.1 Analytic Geometry

Let point $\mathbf{p} = (x, y)$ in two dimensions or $\mathbf{p} = (x, y, z)$ in three dimensions.

The unit vector $\mathbf{u}$ that represents the orientation of a vector $\mathbf{v}$ is the vector of cosines of vector $\mathbf{v}$ with respect to each of the coordinate axes,

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} (\mathbf{v} \cdot \mathbf{e}_1, \mathbf{v} \cdot \mathbf{e}_2, \ldots, \mathbf{v} \cdot \mathbf{e}_n), \tag{A.1}$$

where $\mathbf{e}_i$ is the unit vector for coordinate axis i.

The parametric equation for a line is

$$\mathbf{p}(t) = t\mathbf{u} + \mathbf{p}_0, \tag{A.2}$$

where $\mathbf{u}$ is the unit vector that defines the orientation of the line, $\mathbf{p}_0$ is a point through which the line passes, and $-\infty \leq t \leq \infty$. This equation can also be used to describe rays (half open lines) and line segments by restricting the domain of t. For $0 \leq t \leq \infty$, Equation A.2 represents the vector starting at point $\mathbf{p}_0$ and pointing in the direction $\mathbf{u}$. For $0 \leq t \leq 1$, Equation A.2 represents the unit line segment between point $\mathbf{p}_0$ and point $\mathbf{p}_1$ given by $\mathbf{p}_1 = \mathbf{p}_0 + \mathbf{u}$.

The equation for the line segment between two points can be written as

$$\mathbf{p}(t) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2 \tag{A.3}$$

with $0 \leq t \leq 1$.

Three distinct points $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ define a plane in space. Let $\mathbf{v}_1 = \mathbf{p}_1 - \mathbf{p}_0$ and $\mathbf{v}_2 = \mathbf{p}_2 - \mathbf{p}_0$. If vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ lie in a plane and are not parallel, then the normal to the plane is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2. \tag{A.4}$$

The normal points in the direction of the thumb as the fingers of the right hand sweep from $\mathbf{v}_1$ to $\mathbf{v}_2$. The implicit equation for the plane is the set of points $\mathbf{p}$ that are orthogonal to the plane normal $\mathbf{n}$,

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0, \tag{A.5}$$

where $\mathbf{p}_0$ is a point in the plane and allows the plane to be offset from the origin. If the normal vector $\mathbf{n} = (a, b, c)$, then the plane equation can be written as

$$ax + by + cz + d = 0 \tag{A.6}$$

where $d = -\mathbf{n} \cdot \mathbf{p}_0$ accounts for the displacement of the plane from the origin.

The parametric form of a surface, such as a plane, is an equation of the form $\mathbf{p}(u,v) = (x(u,v), y(u,v), z(u,v))$. The parametric equation for a plane is derived by noting that the origin in the (u,v) domain maps to a point $\mathbf{p}_0 = (x_0, y_0, z_0)$ on the plane, the $\mathbf{u}$-axis in the domain maps to a vector $\mathbf{v}_1$ in the plane, and the $\mathbf{v}$-axis in the domain maps to a vector $\mathbf{v}_2$ in the plane. The parametric equation is

$$\mathbf{p}(u,v) = A \begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{p}_0, \tag{A.7}$$

where the columns of the 3×2 matrix A are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

A.2 Linear Algebra

The notion of a linear space is based on some common assumptions about how physical systems should behave. The power of linear spaces in science and engineering comes from this correspondence between simple mathematical models and real physical systems.

A set of scalars F is a field if the scalars obey the following conditions:

1. If x and y are elements of F, then $x + y$ and xy are elements of F.
2. If x is an element of F, then the additive inverse $-x$ is an element of F.
3. If x is an element of F and $x \neq 0$, then the multiplicative inverse x^{-1} is an element of F.
4. The additive identity 0 and the multiplicative identity 1 are both elements of F.

For example, the set of real numbers with the usual forms of addition and multiplication is a field.

The mathematical model that comes from combining scalars into vectors to represent points in space and many other things is a very powerful model. Let vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Addition of vectors is defined by the scalar addition of the corresponding vector elements,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n), \tag{A.8}$$

and multiplication of a vector $\mathbf{v}$ by a scalar a is defined by applying scalar multiplication to the individual elements of the vector:

$$a\mathbf{v} = (av_1, av_2, \ldots, av_n). \tag{A.9}$$

A vector space, also called a linear space, obeys the following conditions:

1. Addition of vectors $\mathbf{u}$ and $\mathbf{v}$ is commutative

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}. \tag{A.10}$$

2. Addition of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is associative:

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}). \tag{A.11}$$

3. There is a vector denoted by 0 that is the identity element for vector addition:

$$0 + \mathbf{u} = \mathbf{u} + 0 = \mathbf{u}. \tag{A.12}$$

4. For every vector $\mathbf{v}$ there is an additive inverse:

$$\mathbf{v} + (-\mathbf{v}) = 0. \tag{A.13}$$

5. Multiplication of a vector sum by a scalar c is distributed to the individual vectors:

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}. \tag{A.14}$$

6. Multiplication of a vector by a sum of scalars can be rewritten as the sum of the individual scalar multiplications:

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}. \tag{A.15}$$

7. Multiplication of a vector by a product of scalars is associative:

$$(ab)\mathbf{v} = a(b\mathbf{v}). \tag{A.16}$$

8. There is an identity element for multiplication of a vector by a scalar:

$$1\mathbf{u} = \mathbf{u}. \tag{A.17}$$

The linear space is fundamental to science and engineering because it is the mathematical model for systems that behave linearly, which means that the systems behave in a simple way and are easy to understand and use in design. A linear system S obeys the conditions of superposition and homogeneity:

$$S[x + y] = S[x] + S[y] \tag{A.18}$$

$$S[\alpha x] = \alpha S[x] \tag{A.19}$$

which say that the response to a sum of inputs is the sum of the responses to the individual inputs, and the response to an input scaled by a constant is the scaled response. The linearity conditions correspond to our intuitive notions about how things should behave. For example, we expect that a light that is twice as bright would make a scene appear to be twice as bright, and we expect that the result of using two lights should be the sum of the results of using each light alone.

A linear combination is a sum of terms multiplied by constant coefficients:

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n. \tag{A.20}$$

An element $\mathbf{v}$ in an n-dimensional vector space V can be represented as a linear combination of n basis vectors:

$$\mathbf{v} = a_1\mathbf{e}_1 + a_1\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n, \tag{A.21}$$

where $\mathbf{e}_i$ is a basis vector and a_i is the corresponding coefficient. A linear transformation is a mapping between vector spaces and can be implemented by matrix multiplication applied to the vector of coefficients that represent an element in a vector space relative to some basis. The natural basis is the set of vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ with $\mathbf{e}_i = 1$ at position i and zero elsewhere. The coefficients for the natural basis are the coordinates of the vector in the usual Cartesian coordinate system.

Functional analysis extends the notion of vector space to spaces of functions that can be represented by linear combinations of basis functions:

$$f(t) = a_1b_1(t) + a_2b_2(t) + \cdots + a_nb_n(t). \tag{A.22}$$

This is a finite-dimensional vector space, since the basis contains a finite number of basis functions; but there are linear spaces with an infinite number of dimensions that require an infinite number of basis functions, such as Fourier series. Finite-dimensional vector spaces play an important role in statistics, and hence in this book, since a model can be represented by a finite number of parameters that can be estimated by linear regression.

The scalar product supports the notions of the length of a vector and the angle between vectors. The square of the length of a vector is the scalar product of the vector with itself,

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}, \tag{A.23}$$

which is the sum of the squares of the coefficients that represent the vector relative to some basis. The angle between two vectors is the scalar product of the two vectors, normalized by the length of each vector. The angle provides a measure of the difference between two vectors.

Consider two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in the usual Euclidean three-dimensional space that are not collinear. Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ define a plane in space. Any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is a vector in the plane,

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \tag{A.24}$$

and any point in the plane can be reached with a unique linear combination of the vectors. This plane is a subspace of the three-dimensional Euclidean space, and the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis that spans the subspace.

Now consider a third vector $\mathbf{v}_3$ that does not lie in the plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as defined in the preceding paragraph. The three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis that spans the entire three-dimensional space. Consider some vector $\mathbf{v}$ that is not coplanar with $\mathbf{v}_1$ and $\mathbf{v}_2$. Vector $\mathbf{v}$ can be written as a linear combination of two vectors,

$$\mathbf{v} = a_1\mathbf{u} + a_2\mathbf{w}, \tag{A.25}$$

such that vector $\mathbf{u}$ lies in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, and vector $\mathbf{w}$ is perpendicular to that plane. This concept is fundamental to applications in optimization and regression.

The usual scalar product between vectors $\mathbf{u}$ and $\mathbf{v}$ can be generalized to a quadratic form that includes a weight on each scalar product between corresponding vector elements,

$$a_1u_1v_1 + a_2u_2v_2 + \cdots + a_nu_nv_n, \tag{A.26}$$

which can be further generalized to a weighted sum of the products between all pairwise combinations of the elements of vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_iv_j. \tag{A.27}$$

This weighted sum can be written in matrix notation as

$$\mathbf{u}A\mathbf{v}^T, \tag{A.28}$$

where the superscript T denotes the transpose of a row vector into a column vector. There is no loss of generality in assuming that matrix A is symmetric.

The concept of eigenvalues and eigenvectors starts simply with the idea that there may be a linear transformation represented by matrix A and some vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x} \tag{A.29}$$

for some constant λ. Vector $\mathbf{x}$ is an eigenvector of the linear transformation, and λ is its eigenvalue.

Let matrix A be a linear transformation that scales and rotates the natural basis of a vector space so that the axes (basis vectors) correspond in length and orientation with the axes of some ellipsoid that is centered on the origin. Consider a unit vector $\mathbf{u}$ that is already aligned with both an axis of the ellipse and one of the axes of the coordinate system. Since the unit vector is aligned with the axes of the ellipsoid, it will not change orientation under the transformation into the orientation of the ellipsoid, but will change length so that it is scaled to the length of the axis of the ellipsoid; thus,

$$A\mathbf{u} = \lambda\mathbf{u}, \tag{A.30}$$

where λ is the length of the axis of the ellipsoid. This shows that the eigenvalues and eigenvectors are the length and orientation of the axes of the ellipsoid.

A.3 Variational Calculus

Let $f(x, y)$ be the function that is the solution to a variational problem. The general form of a variational problem is

$$\int\int F(x, y, f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}, \ldots)\, dx\, dy.$$

The solution to the variational problem is given as a partial differential equation, called the Euler equation, which is constructed from a formula involving various partial derivatives of F. The general form for a function of two variables is

$$\begin{aligned} F_f - \frac{\partial}{\partial x}F_{f_x} + \frac{\partial^2}{\partial x^2}F_{f_{xx}} - \cdots + (-1)^n \frac{\partial^n}{\partial x^n}F_{f_{x(n)}} \\ - \frac{\partial}{\partial y}F_{f_y} + \frac{\partial^2}{\partial y^2}F_{f_{yy}} - \cdots + (-1)^n \frac{\partial^n}{\partial y^n}F_{f_{y(n)}} = 0 \end{aligned} \tag{A.31}$$

if F does not contain any cross derivatives. In computing the derivatives of the integrand F with respect to a partial derivative of the solution function f, the partial derivative of the solution function is treated as a single variable even though the variable is denoted by a symbol with subscripts. Note how the sign alternates with the order of the derivatives and how the rows of the formula for x and y have the same form. If f is a function of more than two variables, then the Euler equation is extended with an additional sequence of terms for each of the additional variables. If F contains cross derivatives, then there will be additional terms to handle the cross derivatives. If the variational problem requires finding more than one function, then each function will yield another Euler equation.

As an example, consider the problem of determining the surface $z = f(x, y)$ that interpolates a set of data points z_k at locations (x_k, y_k) in a rectangular region R of the image plane, for $k = 1, \ldots, n$. This is an ill-posed problem, since there are an infinite number of functions that can interpolate a set of points. To make the problem well posed, choose the function that is smooth according to the norm

$$\int\int_R \left[\nabla^2 f(x, y)\right]^2 dx\, dy, \tag{A.32}$$

which means choose the function that minimizes Equation A.32 and interpolates the data points z_k at locations (x_k, y_k). Using the calculus of variations, the solution is the biharmonic equation

$$\nabla^4 f(x, y) = 0 \tag{A.33}$$

with boundary conditions

$$f_{yy} = 0 \tag{A.34}$$
$$f_{xxy} = 0 \tag{A.35}$$

along the top and bottom edges of the rectangular domain and boundary conditions

$$f_{xx} = 0 \tag{A.36}$$
$$f_{yyx} = 0 \tag{A.37}$$

along the left and right sides of the rectangular domain. The boundary conditions mean that the interpolated surface does not have to assume any particular value or orientation along the boundary, but should smoothly approach the boundary.

Finding the partial differential equation and boundary conditions for a problem in the variational calculus is only part of the solution. The partial differential equations must be solved by numerical methods.

A.4 Numerical Methods

There are many numerical methods for solving partial differential equations such as the biharmonic Equation A.33 and its boundary conditions. All of the methods involve replacing the partial derivatives with finite difference approximations. There are basically two approaches: (1) replace the partial derivatives in the variational problem, such as Equation A.32, with finite difference approximations to obtain a system of equations that can be solved numerically, or (2) replace the partial derivatives in the partial differential equation derived from the variational problem—Equation A.33, for instance—with finite difference approximations to obtain a system of equations and solve the equations numerically. The numerical methods that solve the variational problem directly can be more efficient, but the iterative methods that solve the partial differential equations are easier to describe and implement.

The simplest finite difference methods solve a partial differential equation at points on a uniform grid. The solution is an array of values for the function $z = f(x, y)$ at grid locations $[i, j]$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The finite difference approximation for the biharmonic evaluated at grid location $[i, j]$ is a linear combination of the function values at neighboring grid locations. Imagine that the n by m grid of function values $f[i, j]$ is unfolded into a long vector $f[k]$, with $k = (i - 1)m + j$. Let N_k be the list of offsets for the grid locations of the neighbors of $f[k]$ in the grid. The biharmonic equation is approximated at each grid location by

$$\nabla^4 f(x, y) \approx a_0 f[k] + \sum_{l \in N_k} a_l f[k + l] = 0 \tag{A.38}$$

with changes to the coefficients near the boundaries of the domain. For each grid location, this linear equation provides one row in a system of linear equations

$$Af = 0 \tag{A.39}$$

except at the grid locations where a known value is being interpolated, in which case that row is filled with zeros except at $f[k] = 1$ and the right-hand size is the known value z_k. The system of linear equations can be solved with sparse matrix techniques, or Equation A.38 can be solved for $f[k]$ and this formula repeated over all grid locations that do not have a known value until the solution vector does not change significantly. This is the method of successive approximation and is described in *Numerical Recipes* [197], along with more sophisticated methods for solving partial differential equations and variational problems.

Further Reading

The series of books *Graphics Gems* [89] provide many useful formulas and algorithms for geometry.

Linear spaces are widely used in science and engineering. Lang [153] provides a rigorous introduction to linear spaces. Noble and Daniel [187] cover linear spaces with many practical applications. Naylor and Sell [182] cover functional analysis, which is the extension of vector spaces to spaces of functions, and include many examples from science and engineering.

Many problems can be formulated as optimization problems and solved using the variational calculus. An excellent introduction to the variational calculus is provided by Courant and Hilbert [62, Chap. 4]. The variational calculus produces a partial differential equation that usually must be solved by numerical methods. The book *Numerical Recipes* is an excellent introduction to numerical methods [197].