CDCL SAT Solvers & SAT-Based Problem Solving

Joao Marques-Silva[1,2] & Mikolas Janota[2]

[1]University College Dublin, Ireland

[2]IST/INESC-ID, Lisbon, Portugal

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Aalto University, Espoo, Finland

The Success of SAT

- Well-known NP-complete decision problem [C71]
- In practice, SAT is a success story of Computer Science
  - Hundreds (even more?) of practical applications

Part I

CDCL SAT Solvers

Outline

Outline

Basic Definitions

DPLL Solvers

CDCL Solvers

What Next in CDCL Solvers?

Preliminaries

- Variables: $w, x, y, z, a, b, c, \ldots$
- Literals: $w, \bar{x}, \bar{y}, a, \ldots$, but also $\neg w, \neg y, \ldots$
- Clauses: disjunction of literals or set of literals
- Formula: conjunction of clauses or set of clauses
- Model (satisfying assignment): partial/total mapping from variables to $\{0, 1\}$
- Formula can be SAT/UNSAT

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- Clauses: disjunction of literals or set of literals
- Formula: conjunction of clauses or set of clauses
- Model (satisfying assignment): partial/total mapping from variables to $\{0, 1\}$
- Formula can be SAT/UNSAT
- Example:

$$\mathcal{F} \triangleq (r) \wedge (\bar{r} \vee s) \wedge (\bar{w} \vee a) \wedge (\bar{x} \vee b) \wedge (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)$$

  - Example models:
    - $\{r, s, a, b, c, d\}$
    - $\{r, s, \bar{x}, y, \bar{w}, z, \bar{a}, b, c, d\}$

Resolution

- Resolution rule: [DP60,R65]

$$\frac{(\alpha \vee x) \qquad\qquad\qquad (\beta \vee \bar{x})}{(\alpha \vee \beta)}$$

  - Complete proof system for propositional logic

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$$\begin{tikzcd}
(x \vee a) \arrow[d, dash] & & (\bar{x} \vee a) \arrow[dll, dash] & & (\bar{y} \vee \bar{a}) \arrow[drr, dash] & & (y \vee \bar{a}) \arrow[d, dash] \\
(a) \arrow[drrr, dash] & & & & & & (\bar{a}) \arrow[dlll, dash] \\
& & & \perp & & &
\end{tikzcd}$$

  - Extensively used with (CDCL) SAT solvers

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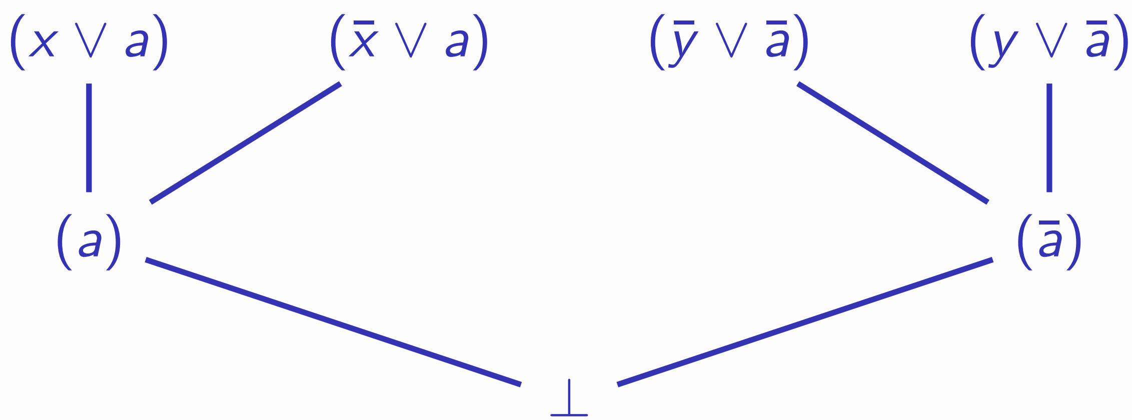
  - Complete proof system for propositional logic



  - Extensively used with (CDCL) SAT solvers

- Self-subsuming resolution (with $\alpha' \subseteq \alpha$): [e.g. SP04,EB05]

$$\frac{(\alpha \vee x) \qquad\qquad (\alpha' \vee \bar{x})}{(\alpha)}$$

  - (α) subsumes $(\alpha \vee x)$

Unit Propagation

$$
\begin{aligned}
\mathcal{F} \quad = \quad & (r) \wedge (\bar{r} \vee s) \wedge \\
& (\bar{w} \vee a) \wedge (\bar{x} \vee \bar{a} \vee b) \\
& (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)
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- Decisions / Variable Branchings:
  $w = 1, x = 1, y = 1, z = 1$

Unit Propagation

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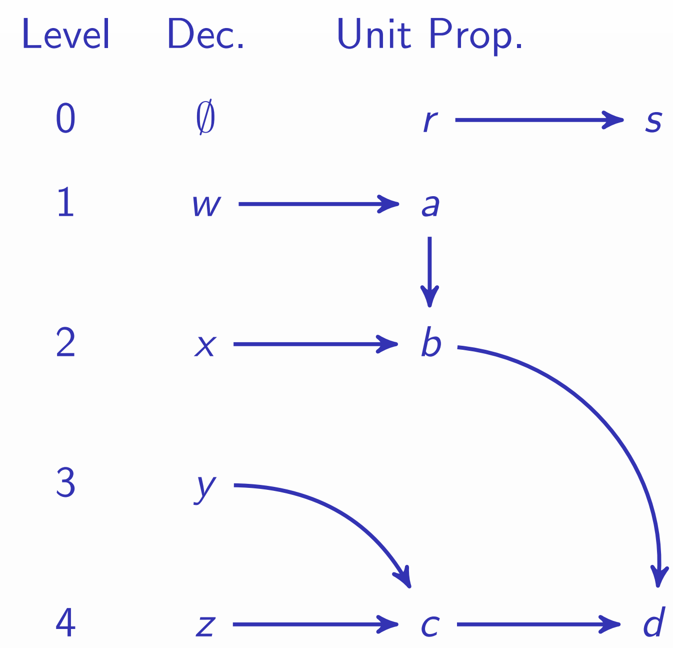

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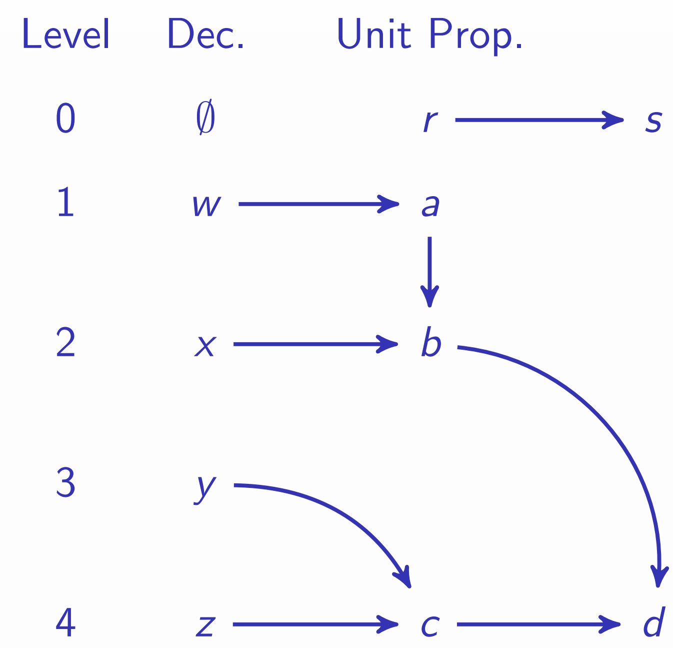

- Decisions / Variable Branchings:
  $w = 1, x = 1, y = 1, z = 1$

- Additional definitions:
  - Antecedent (or reason) of an implied assignment
    - $(\bar{b} \vee \bar{c} \vee d)$ for d
  - Associate assignment with decision levels
    - $w = 1 @ 1,\ x = 1 @ 2,\ y = 1 @ 3,\ z = 1 @ 4$
    - $r = 1 @ 0,\ d = 1 @ 4,\ \ldots$

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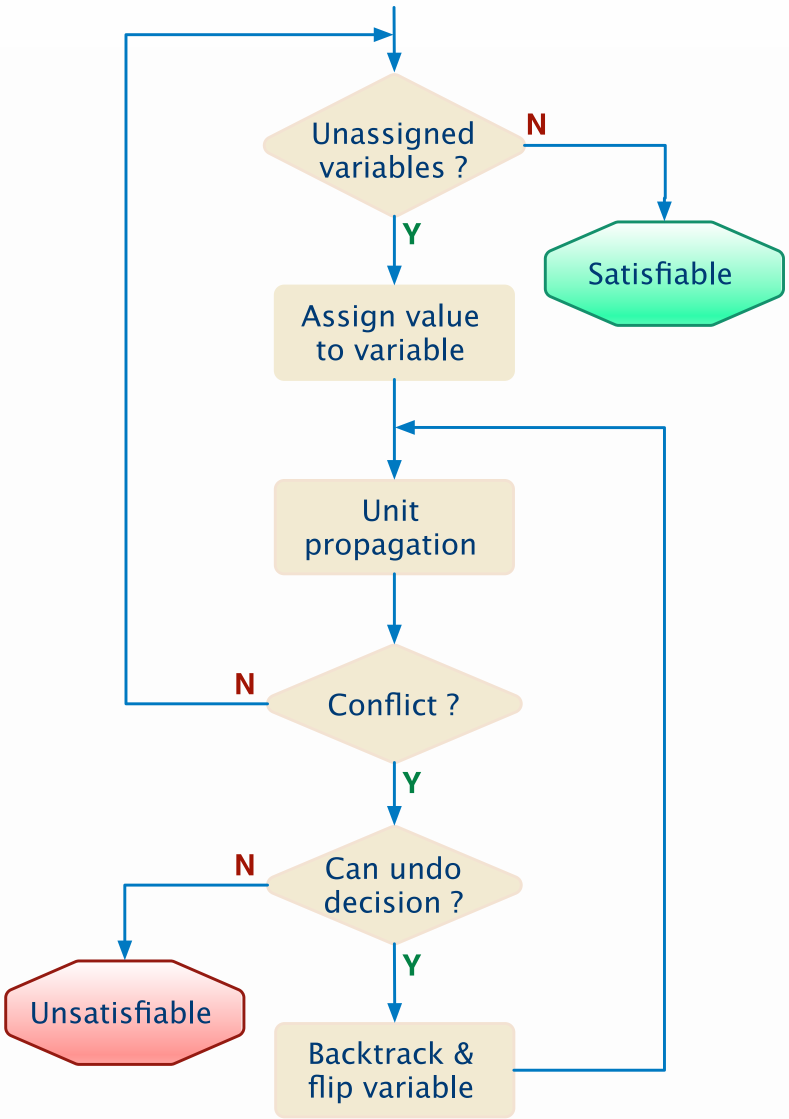
CDCL Solvers

What Next in CDCL Solvers?

The DPLL Algorithm



- Optional: pure literal rule

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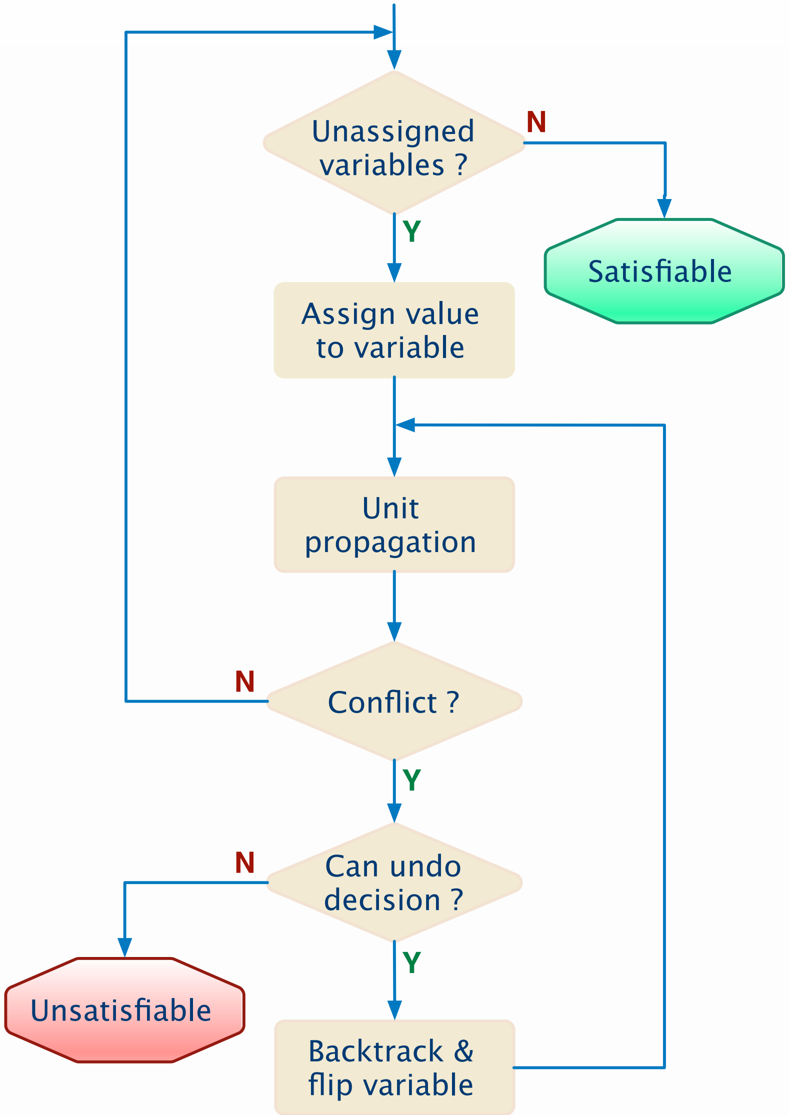

$$\mathcal{F} = (x \vee y) \wedge (a \vee b) \wedge (\bar{a} \vee b) \wedge (a \vee \bar{b}) \wedge (\bar{a} \vee \bar{b})$$

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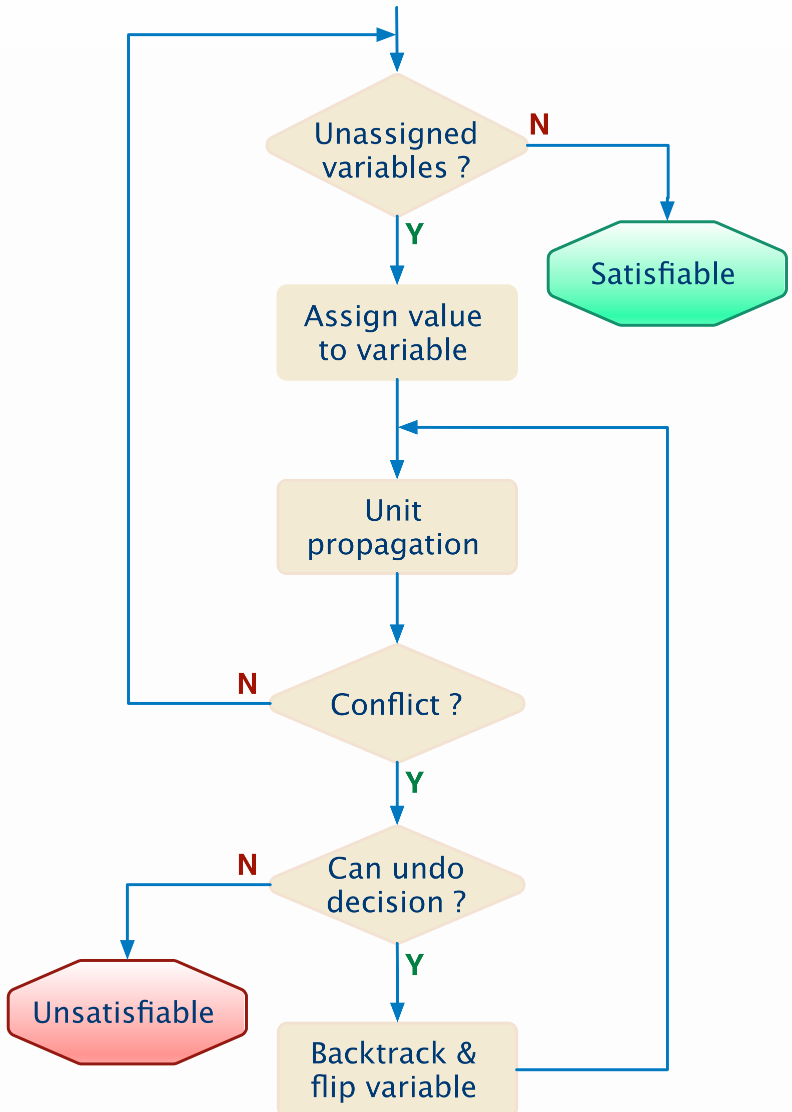

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Level	Dec.	Unit Prop.
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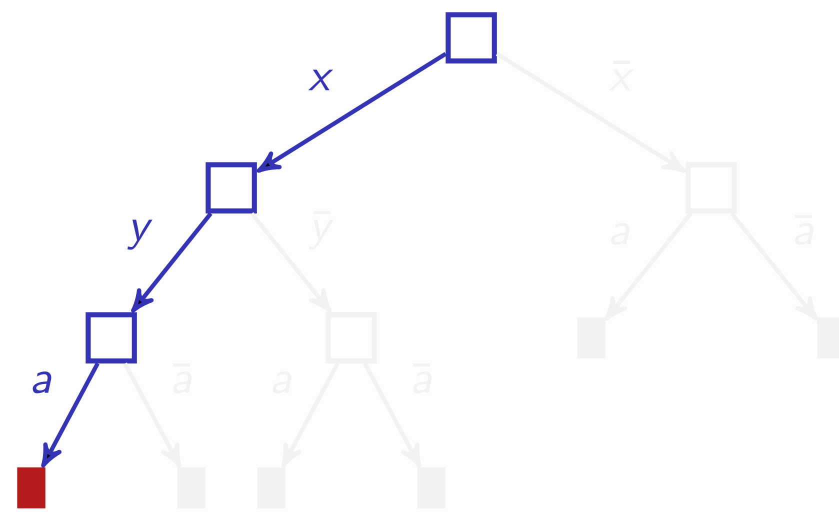
1	x	
2	y	
3	a	$\rightarrow b \rightarrow \perp$



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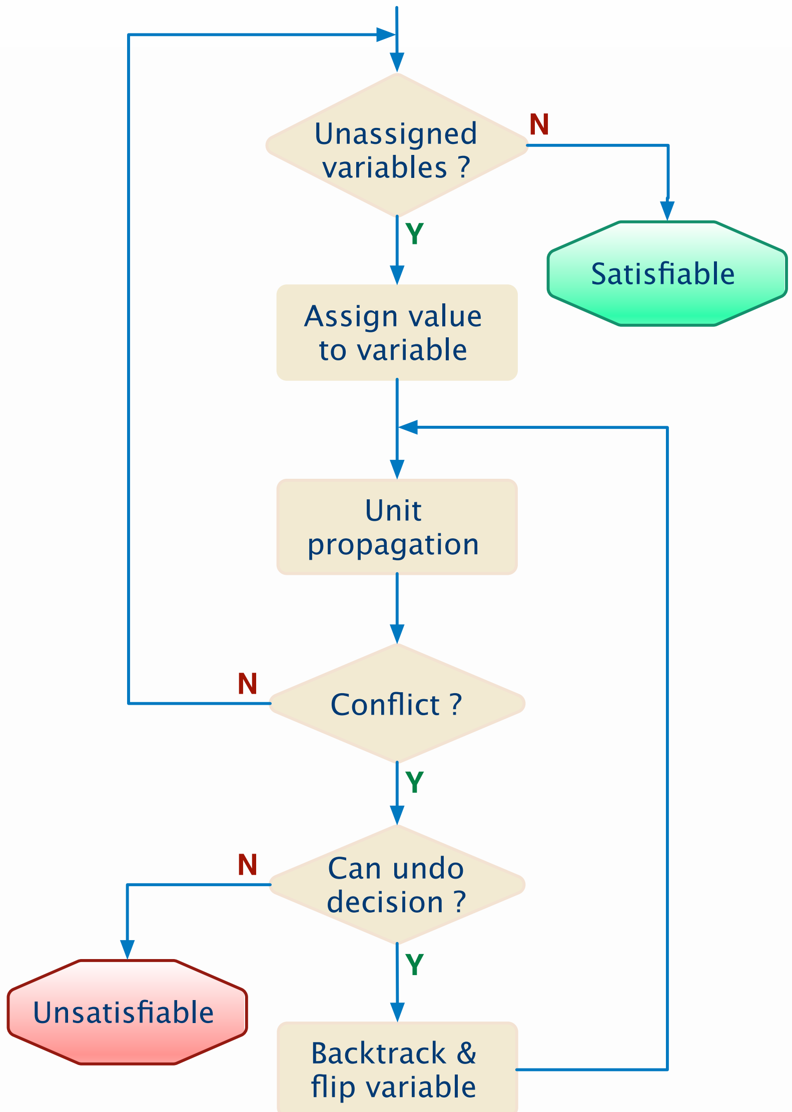

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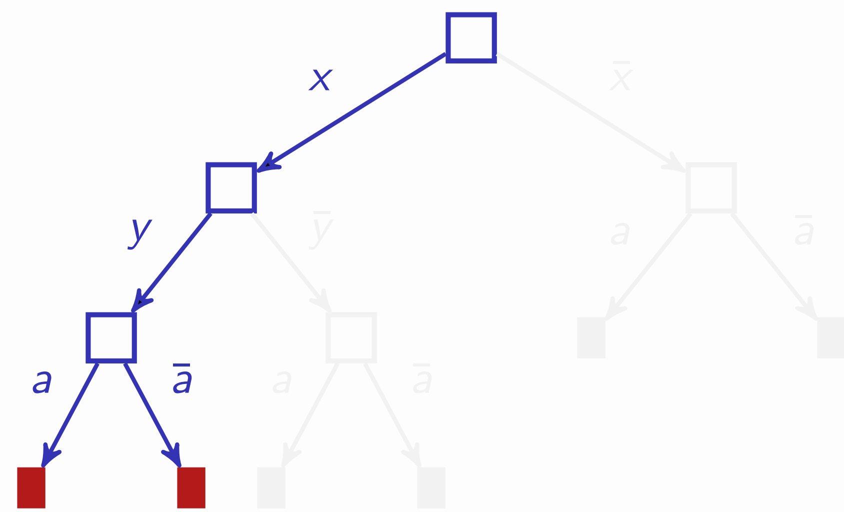

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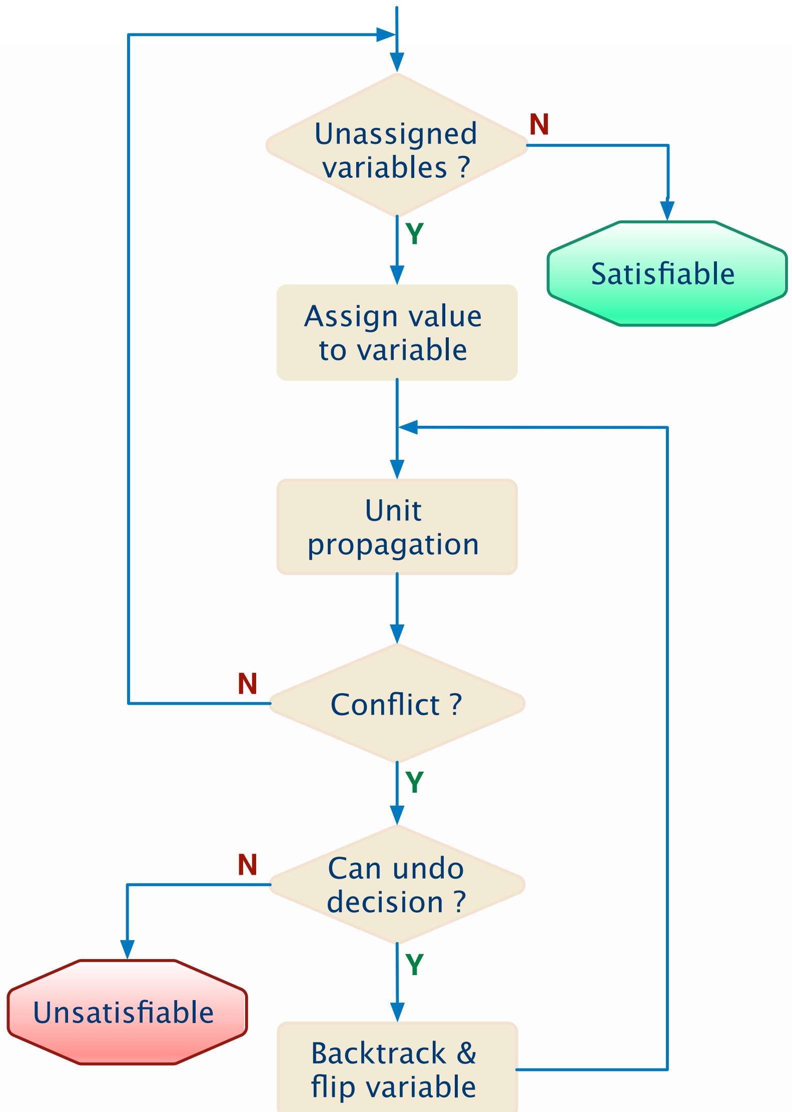

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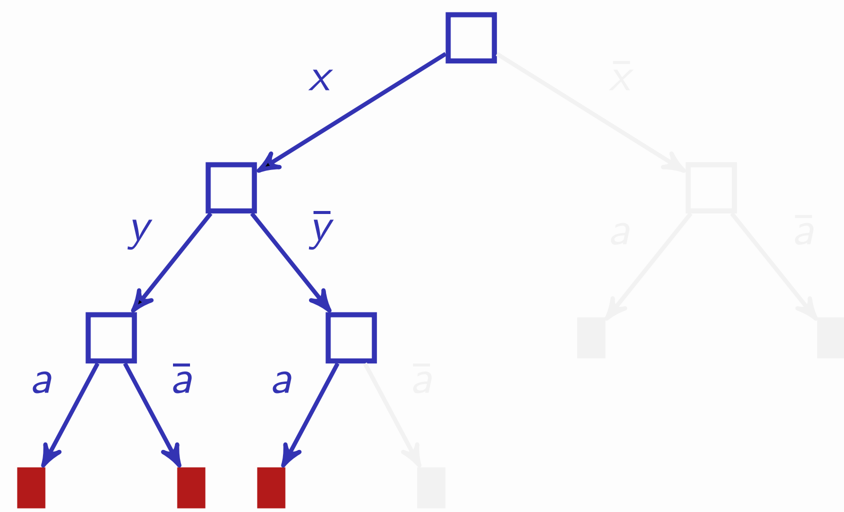

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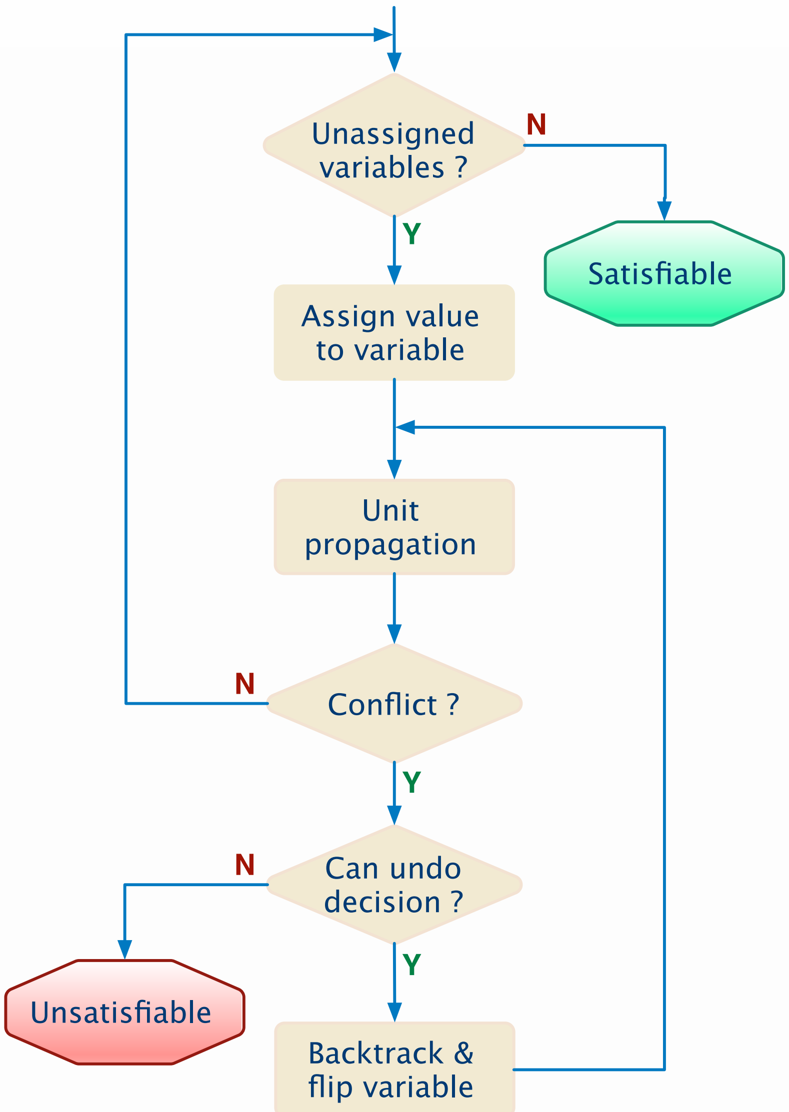

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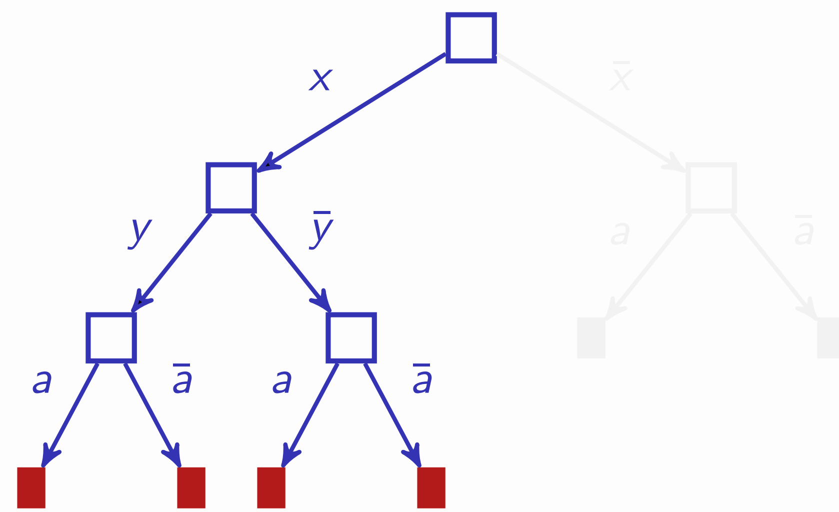

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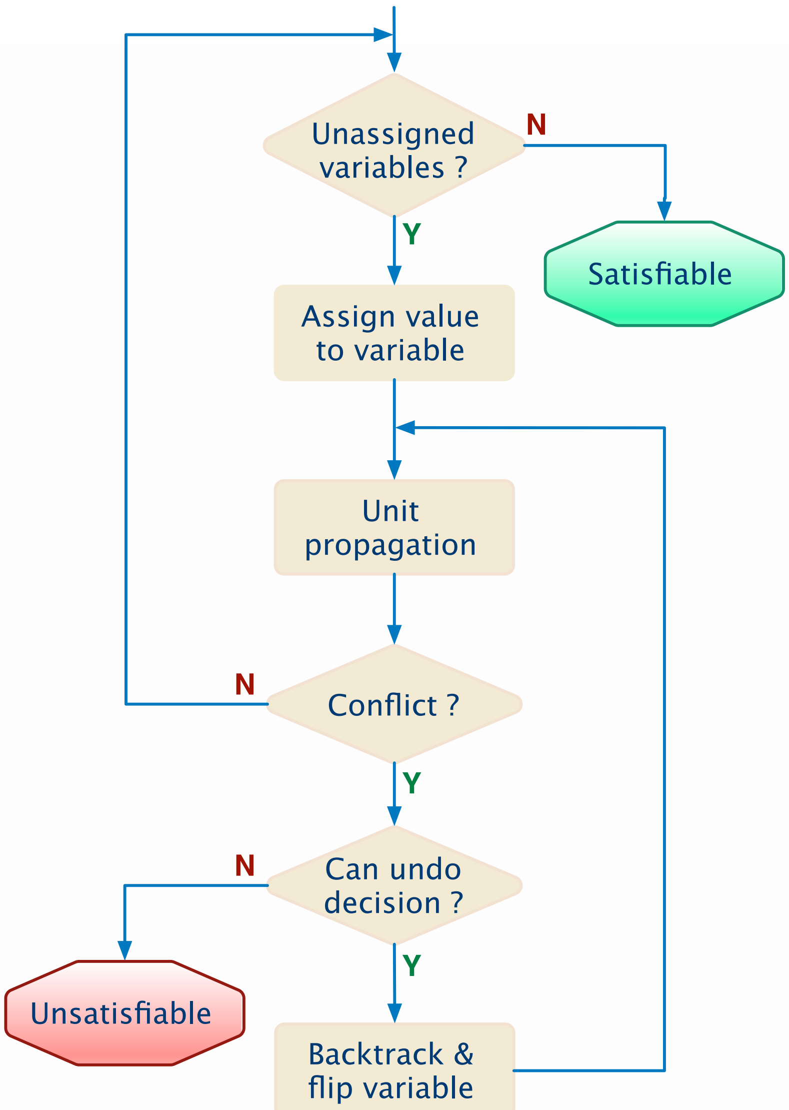

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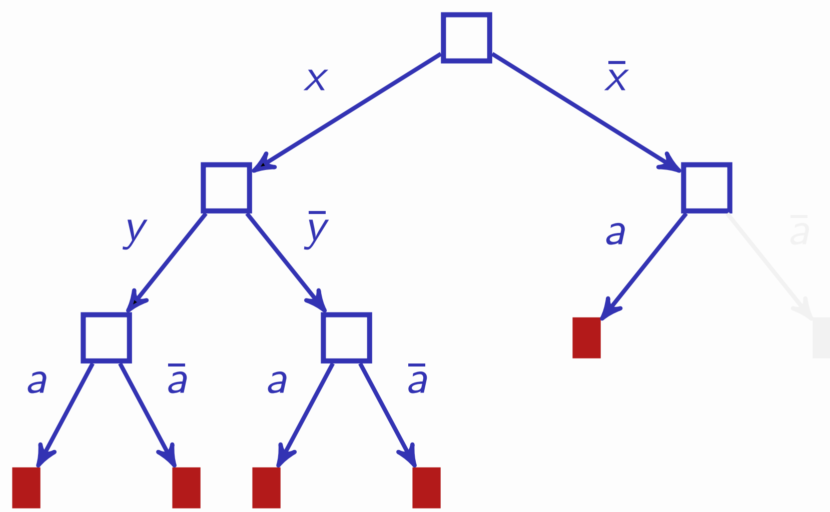

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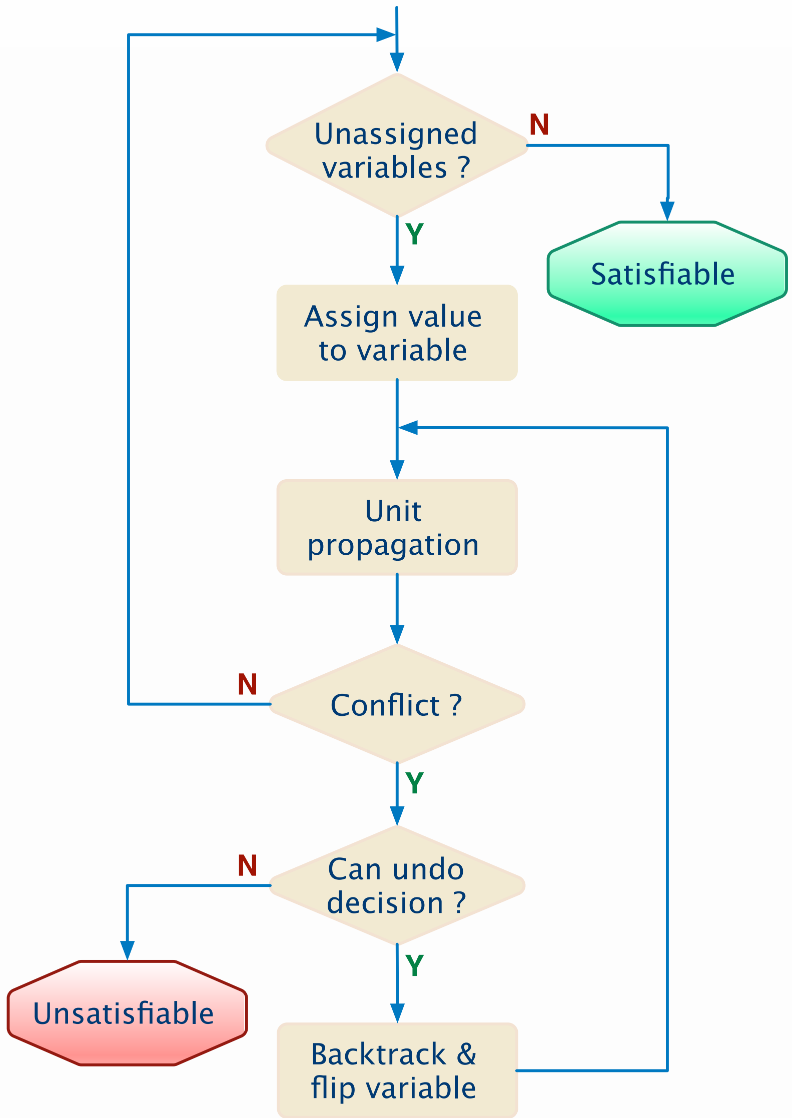

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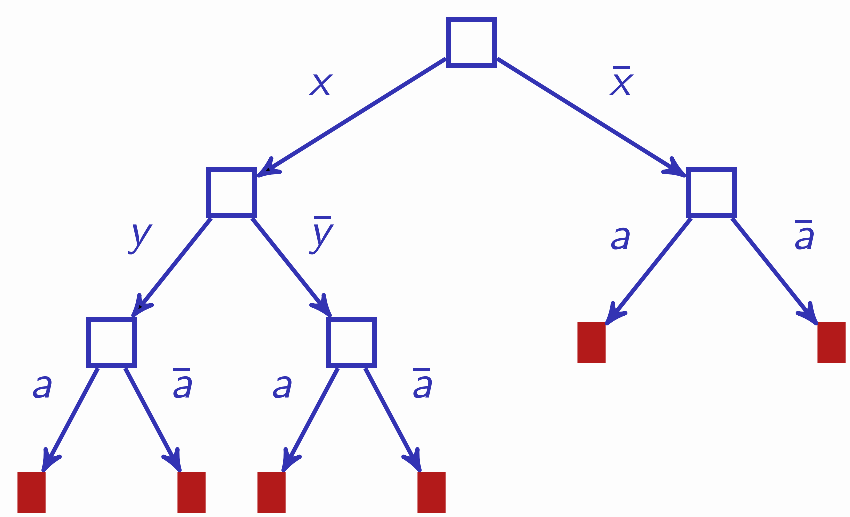

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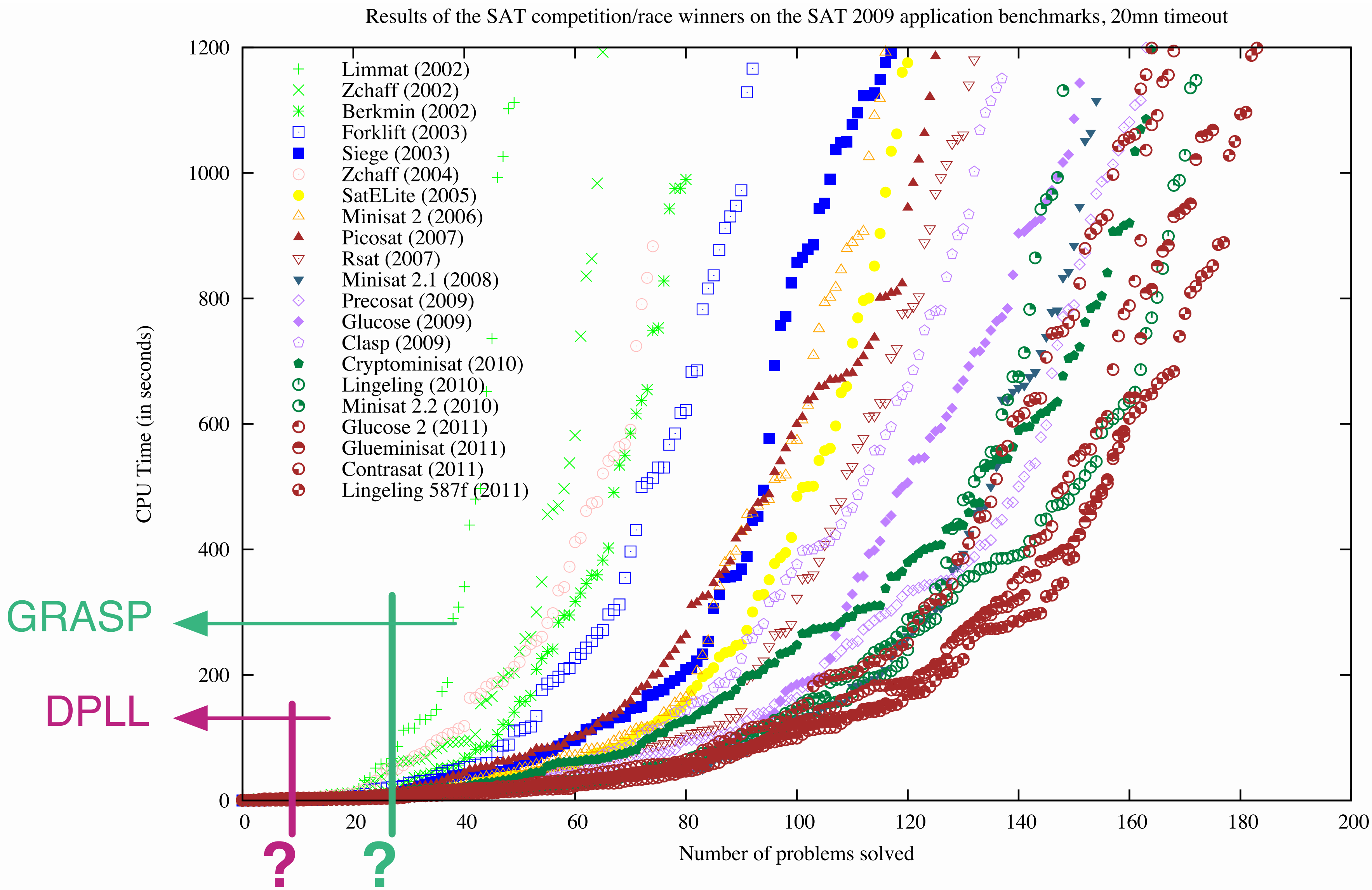
Outline

What is a CDCL SAT Solver?

- Extend DPLL SAT solver with: [DP60,DLL62]
  - Clause learning & non-chronological backtracking [MSS96,BS97,Z97]
    - Exploit UIPs [MSS96,SSS12]
    - Minimize learned clauses [SB09,VG09]
    - Opportunistically delete clauses [MSS96,MSS99,GN02]
  - Search restarts [GSK98,BMS00,H07,B08]
  - Lazy data structures
    - Watched literals [MMZZM01]
  - Conflict-guided branching
    - Lightweight branching heuristics [MMZZM01]
    - Phase saving [PD07]
  - ...

How Significant are CDCL SAT Solvers?



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CDCL Solvers

- Clause Learning, UIPs & Minimization
- Search Restarts & Lazy Data Structures

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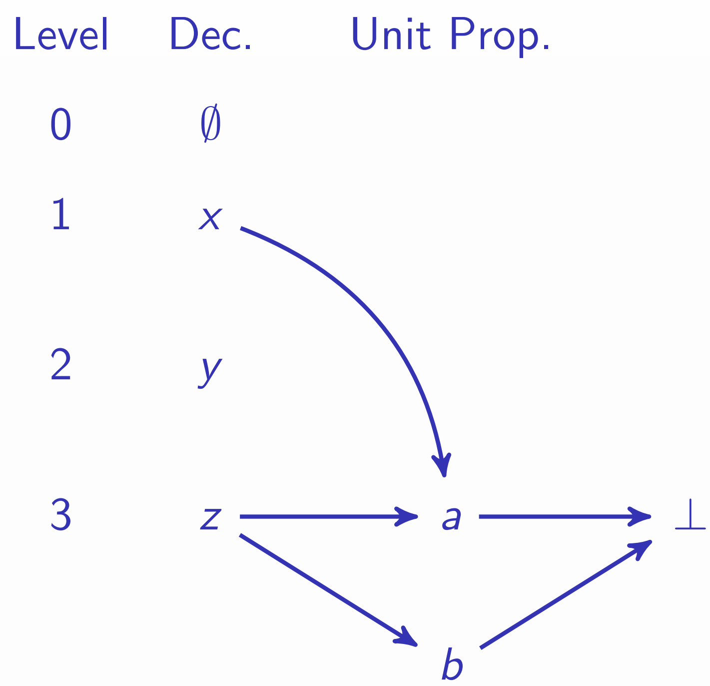
Clause Learning



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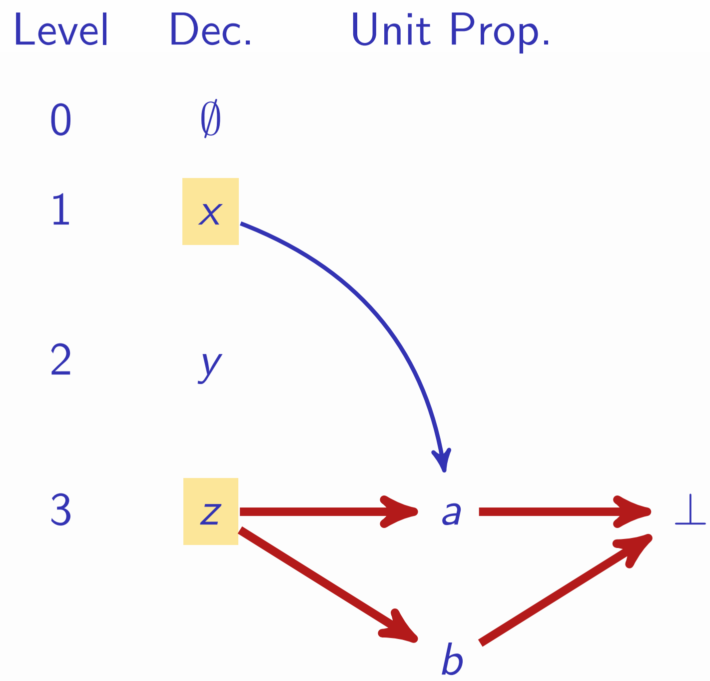


$(\bar{a} \vee \bar{b})$ $(\bar{z} \vee b)$ $(\bar{x} \vee \bar{z} \vee a)$

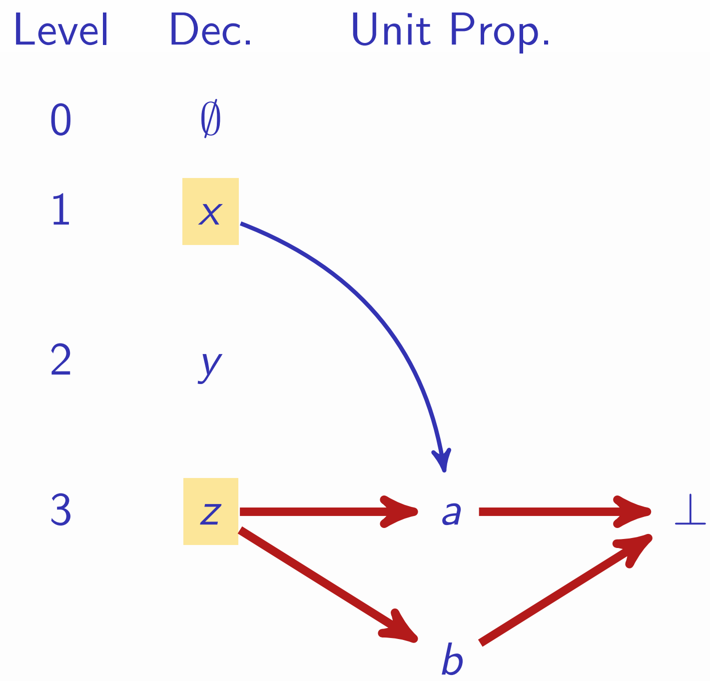
ab -> false = !(ab) + false = !a + !b

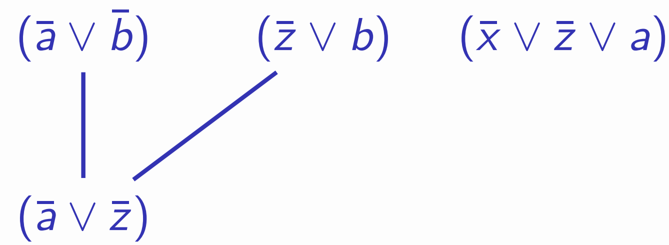
z -> b = !z + b

xz -> a. =. !(xz) + a =. !x + !z + a

- Analyze conflict
  - Reasons: x and z
    - Decision variable & literals assigned at lower decision levels
  - Create **new** clause: $(\bar{x} \vee \bar{z})$
- Can relate clause learning with resolution

Clause Learning





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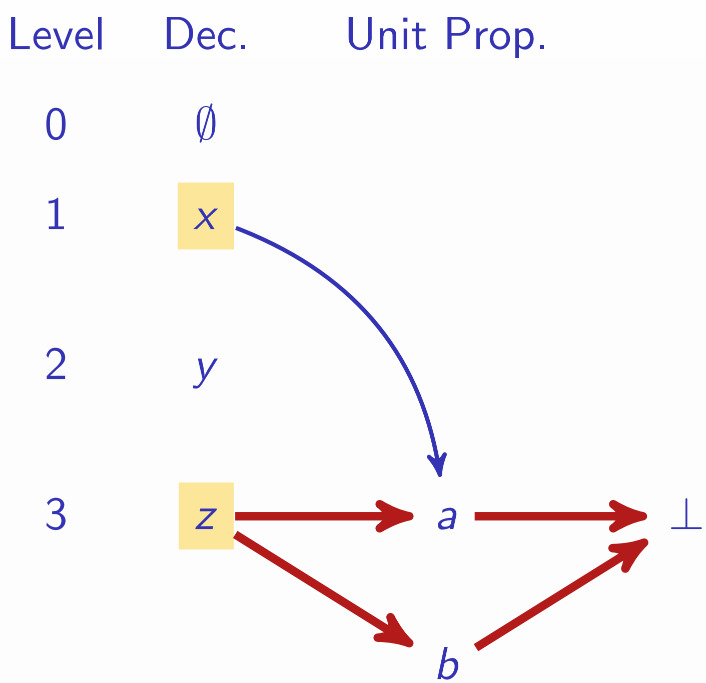

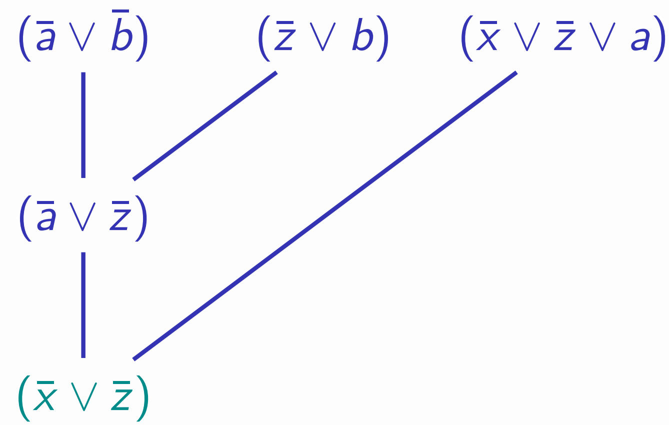

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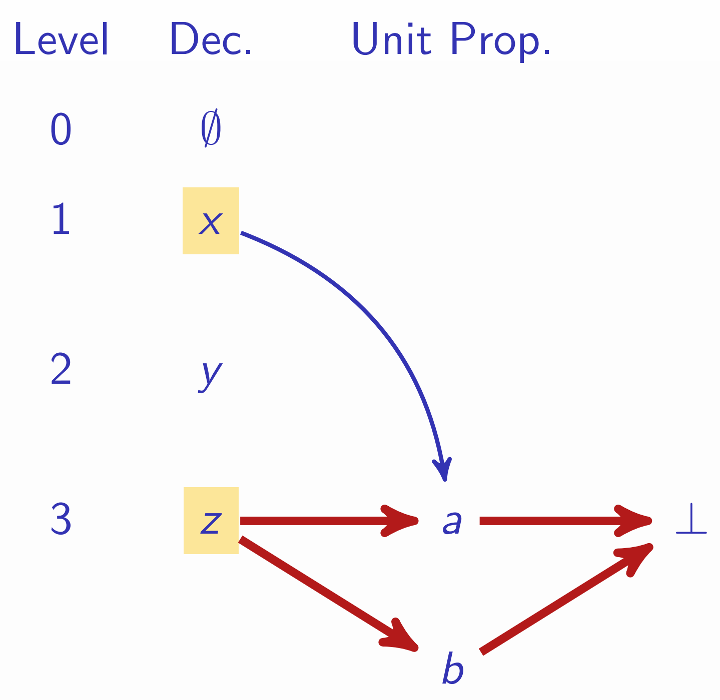
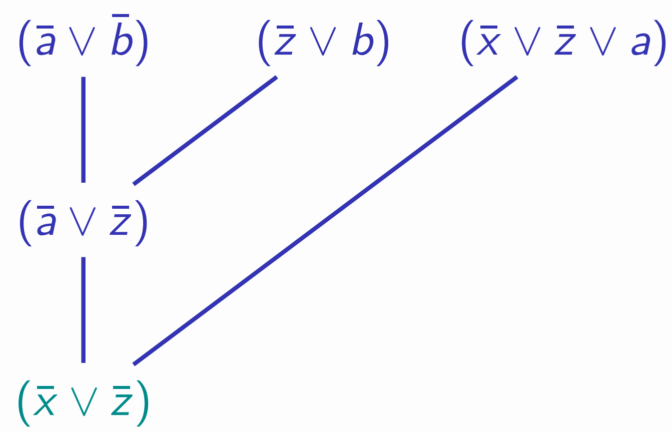
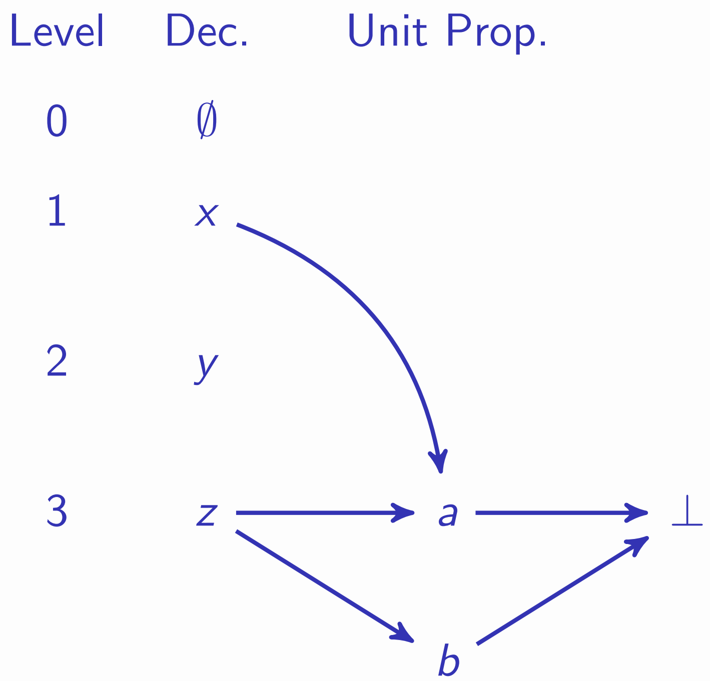




- Analyze conflict
  - Reasons: x and z
    - Decision variable & literals assigned at lower decision levels
  - Create **new** clause: $(\bar{x} \lor \bar{z})$
- Can relate clause learning with resolution
  - Learned clauses result from (**selected**) resolution operations

Clause Learning – After Bracktracking



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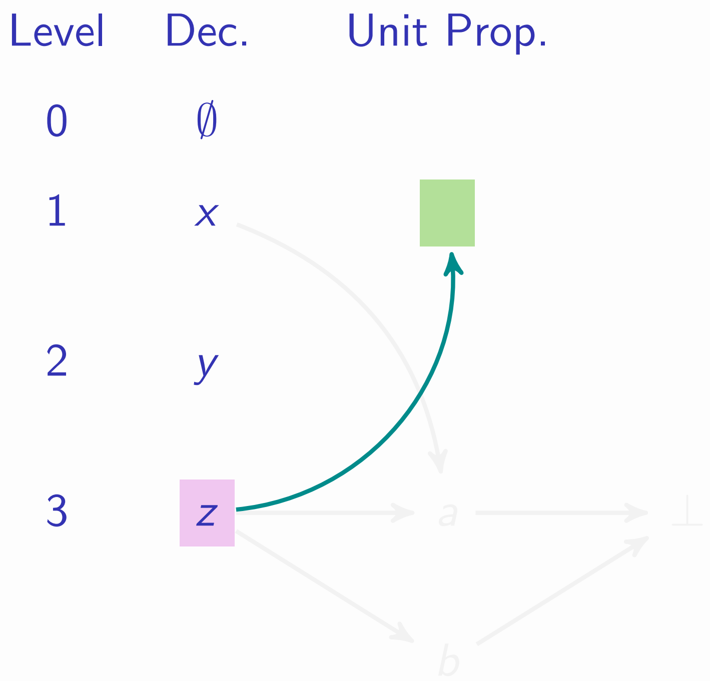

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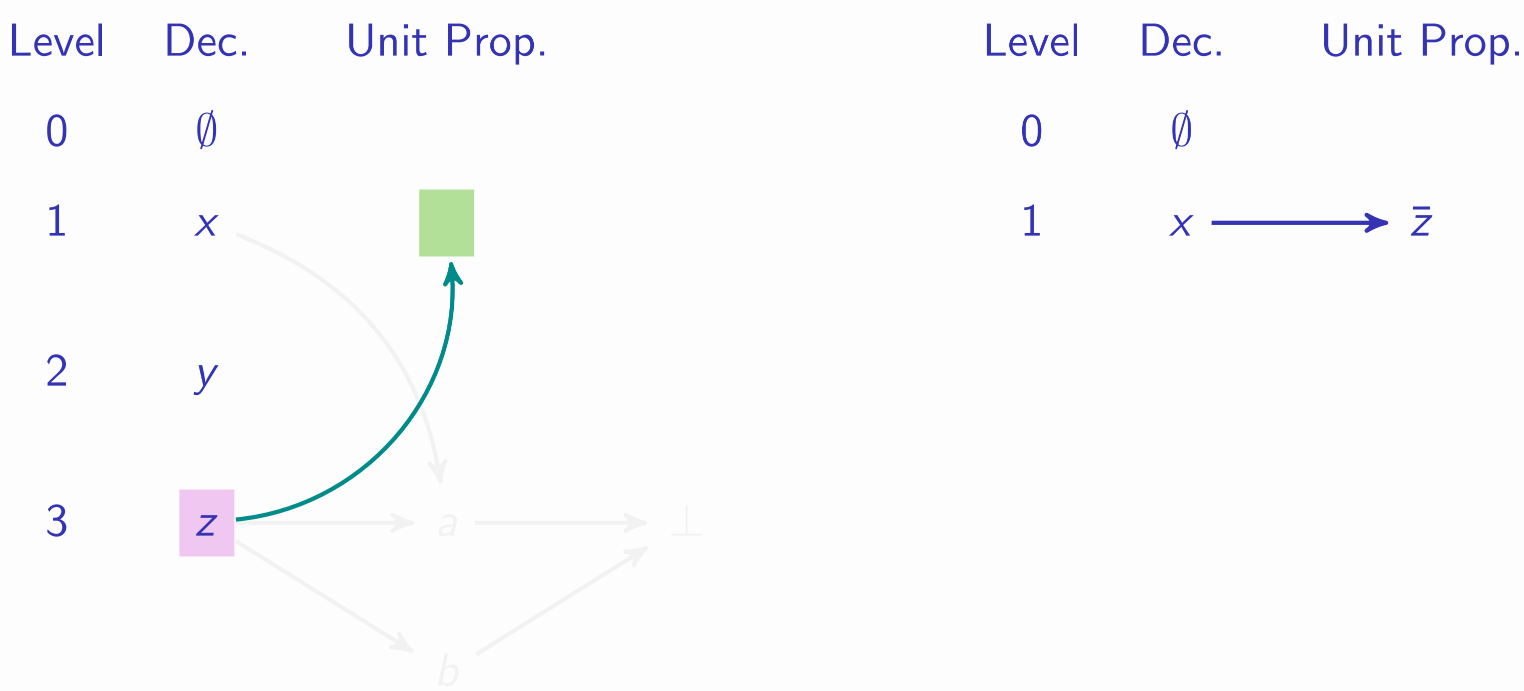

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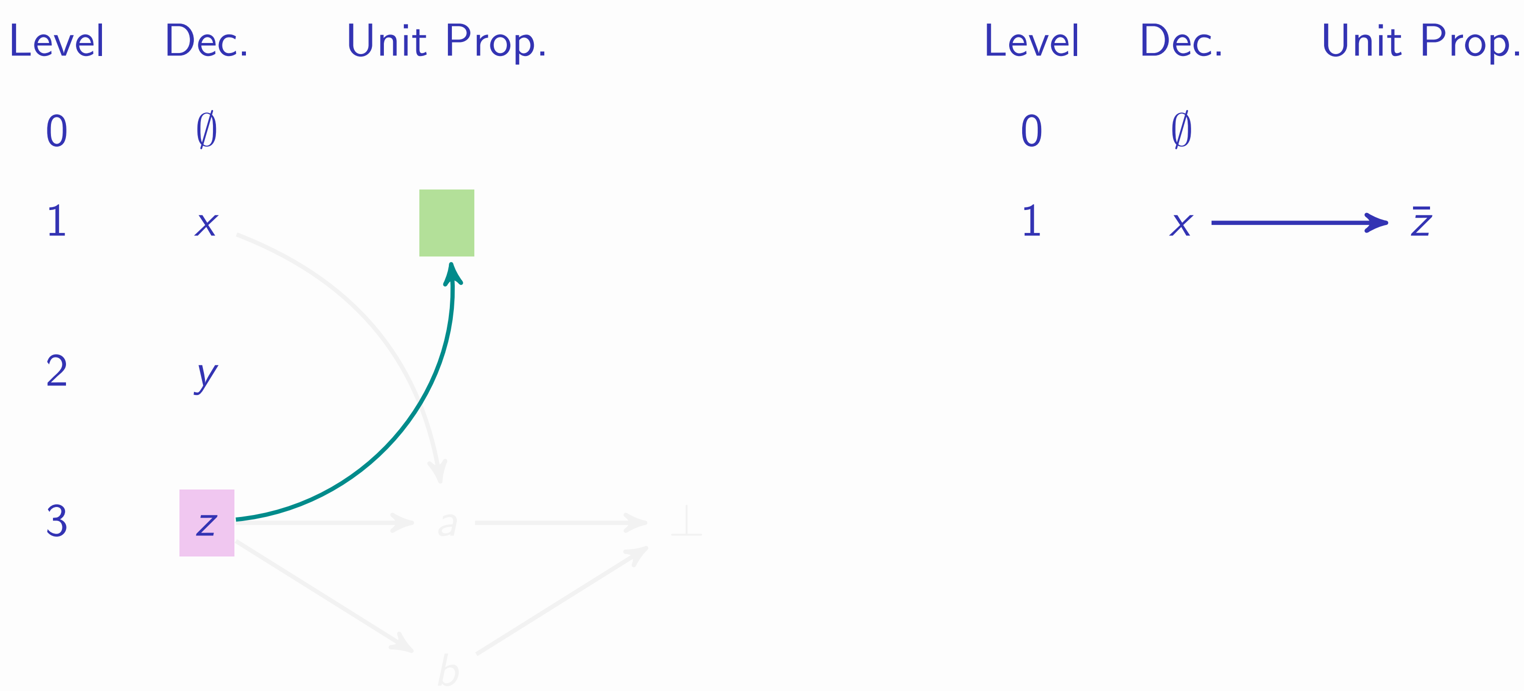
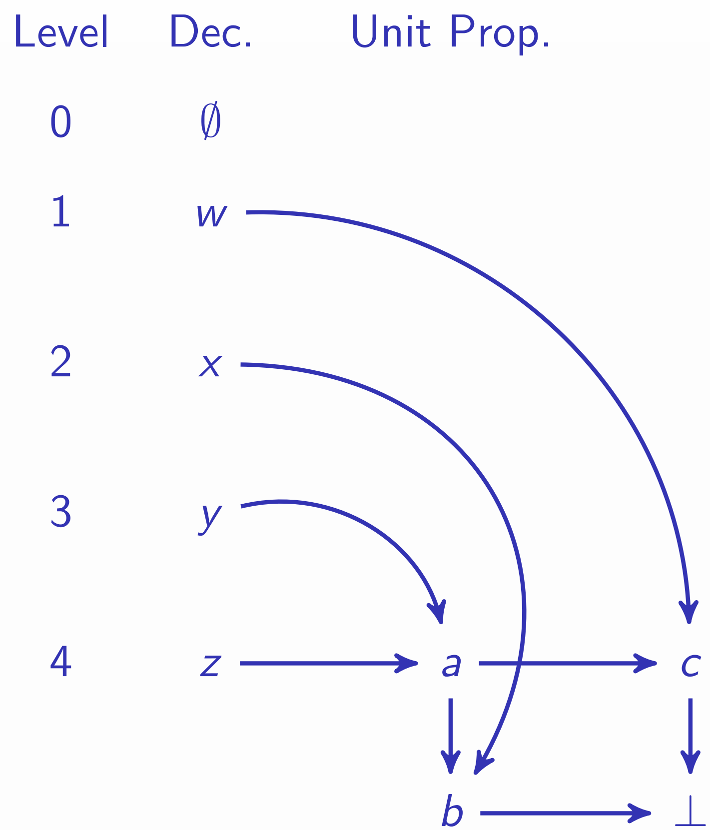


- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1
- Learned clauses are always asserting [MSS96,MSS99]
- Backtracking differs from plain DPLL:
  - Always bactrack after a conflict [MMZZM01]

Unique Implication Points (UIPs)



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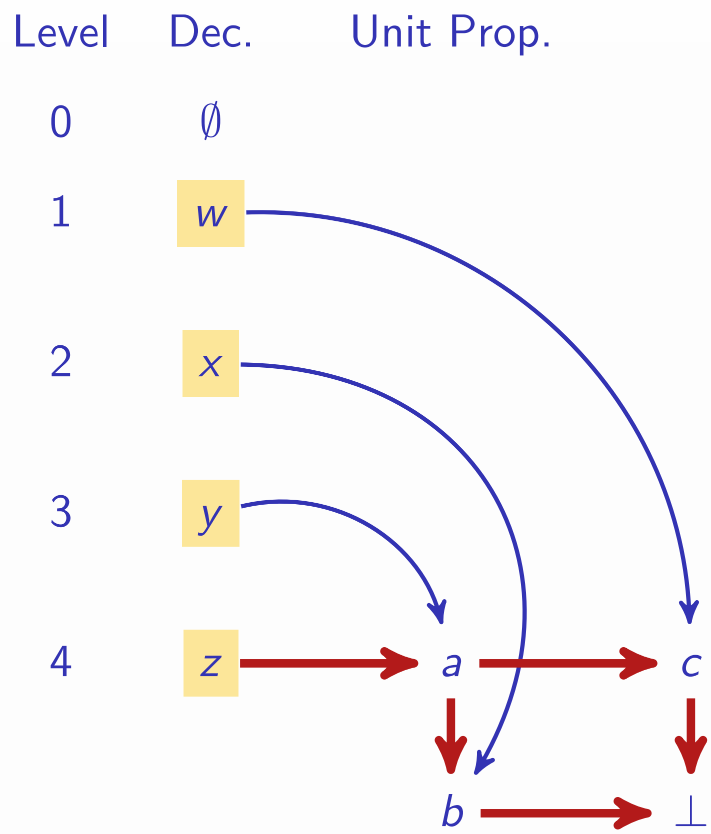

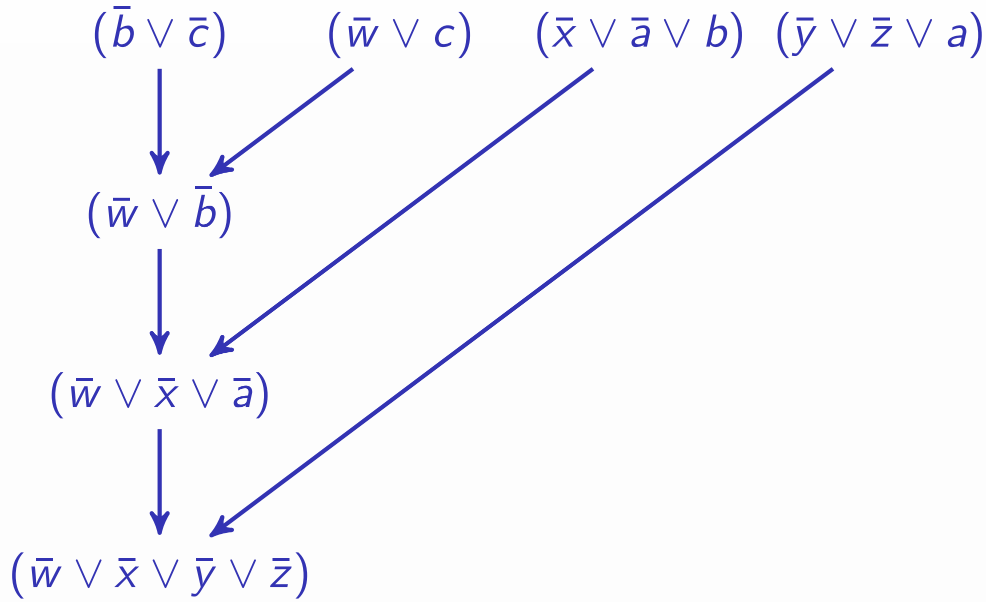

- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

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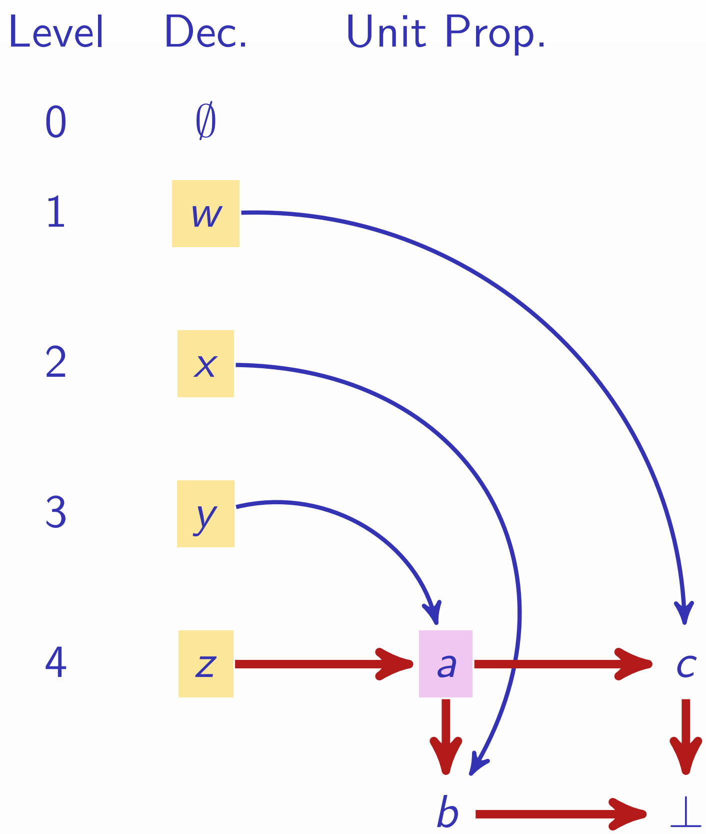

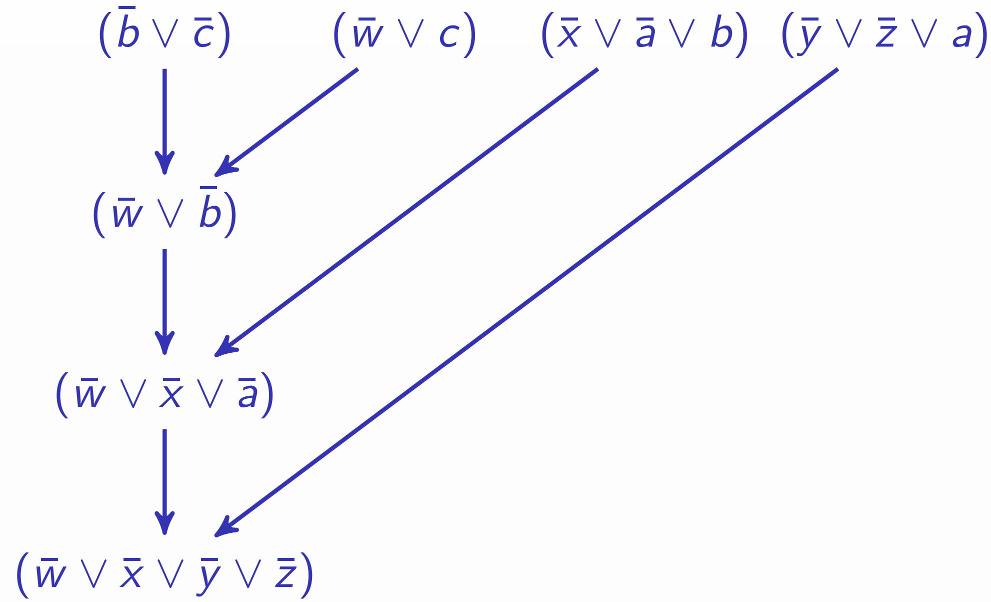

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- But a is an UIP

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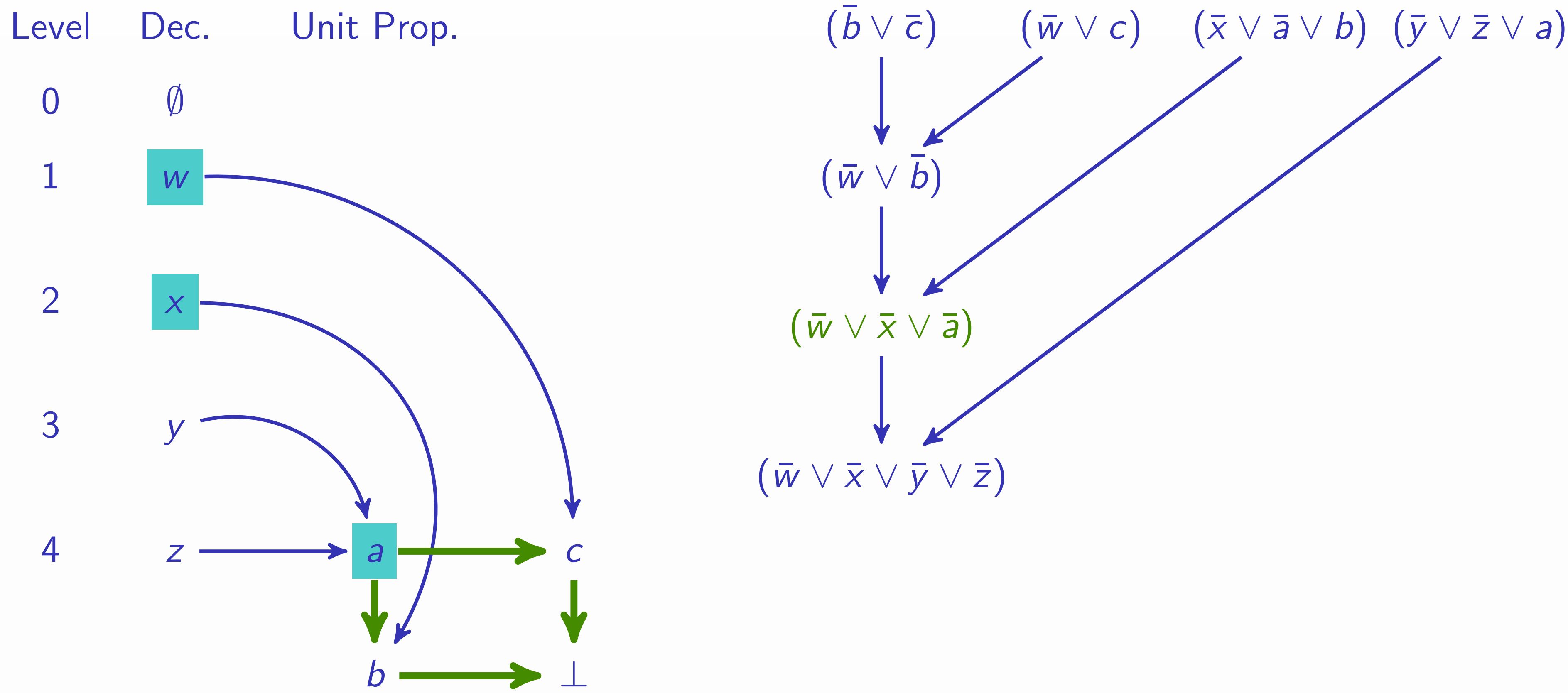
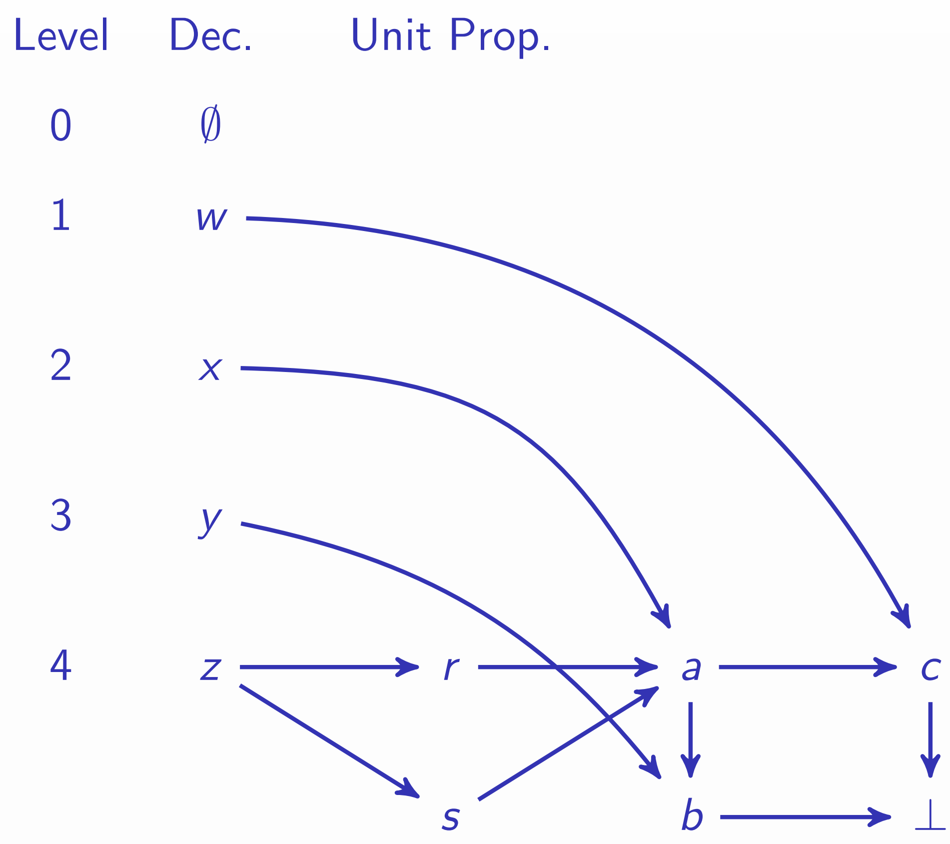


- ~~Learn clause $(\bar{w} \lor \bar{x} \lor \bar{y} \lor \bar{z})$~~
- But a is an UIP
- Learn clause $(\bar{w} \lor \bar{x} \lor \bar{a})$

Multiple UIPs



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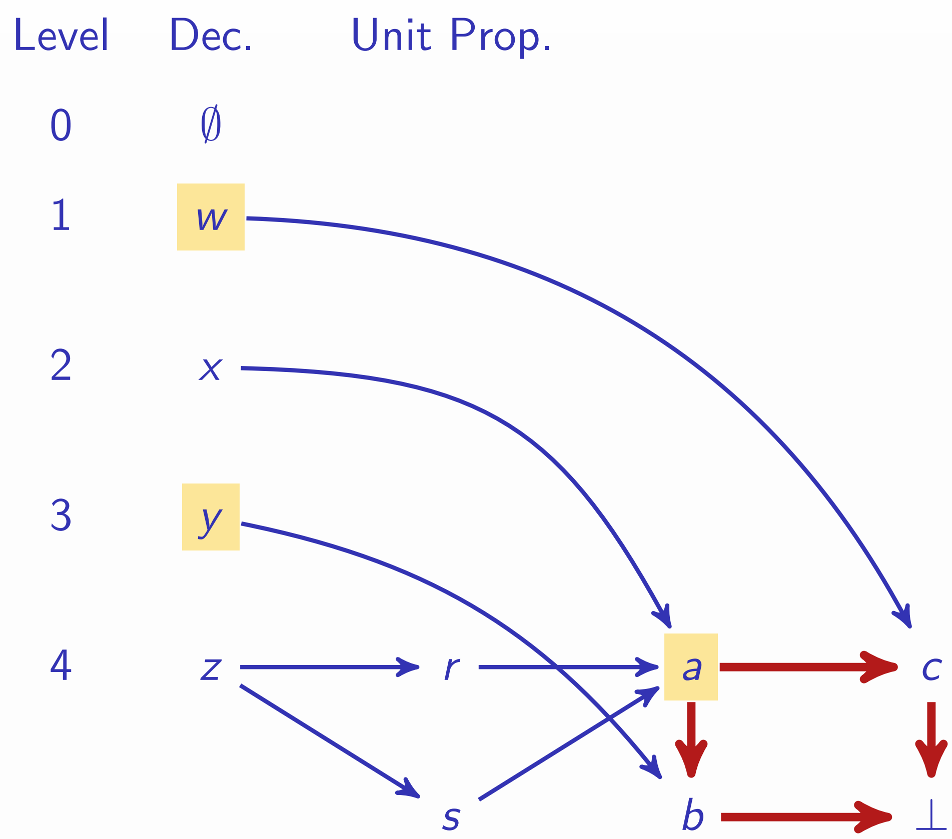

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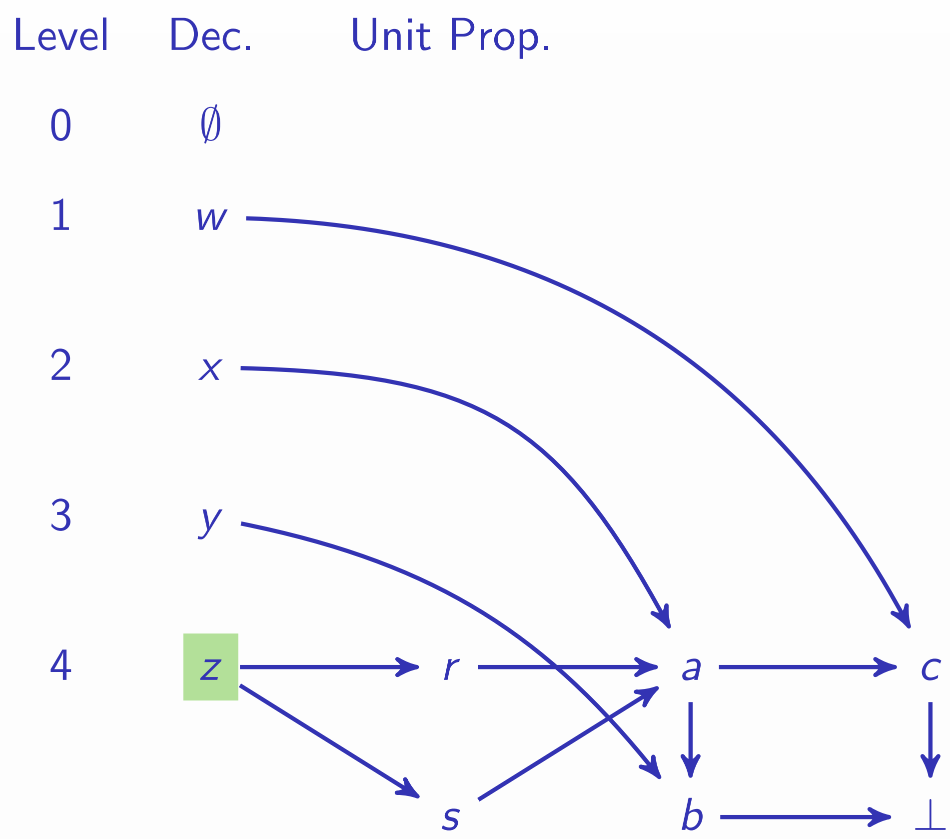

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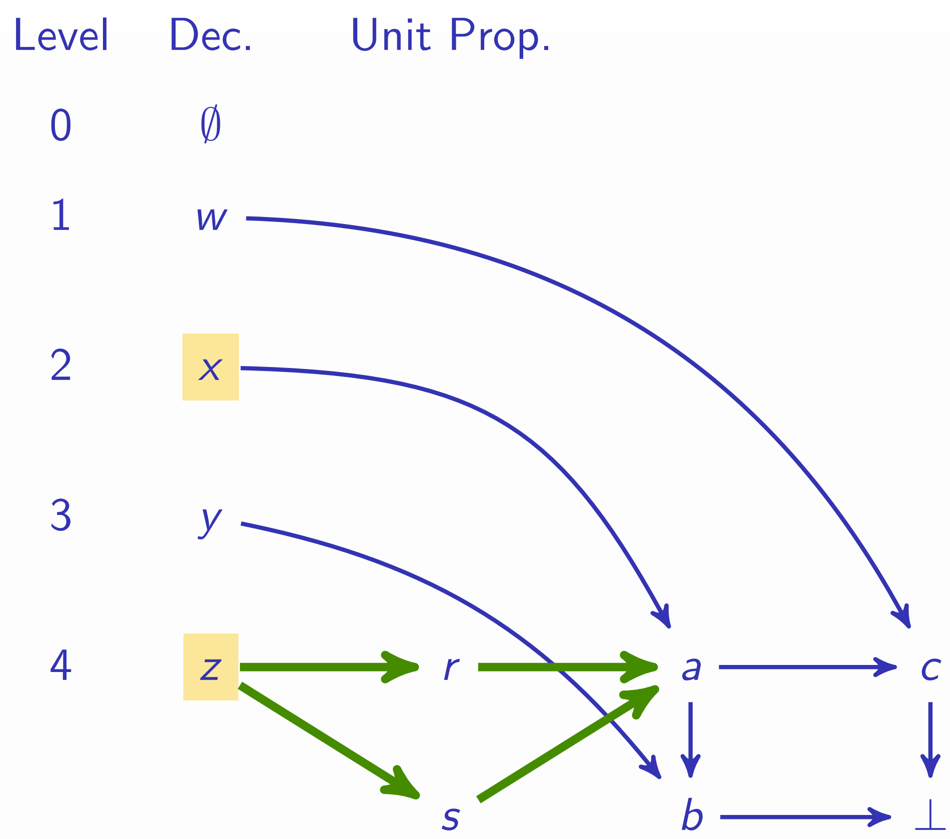

- First UIP:
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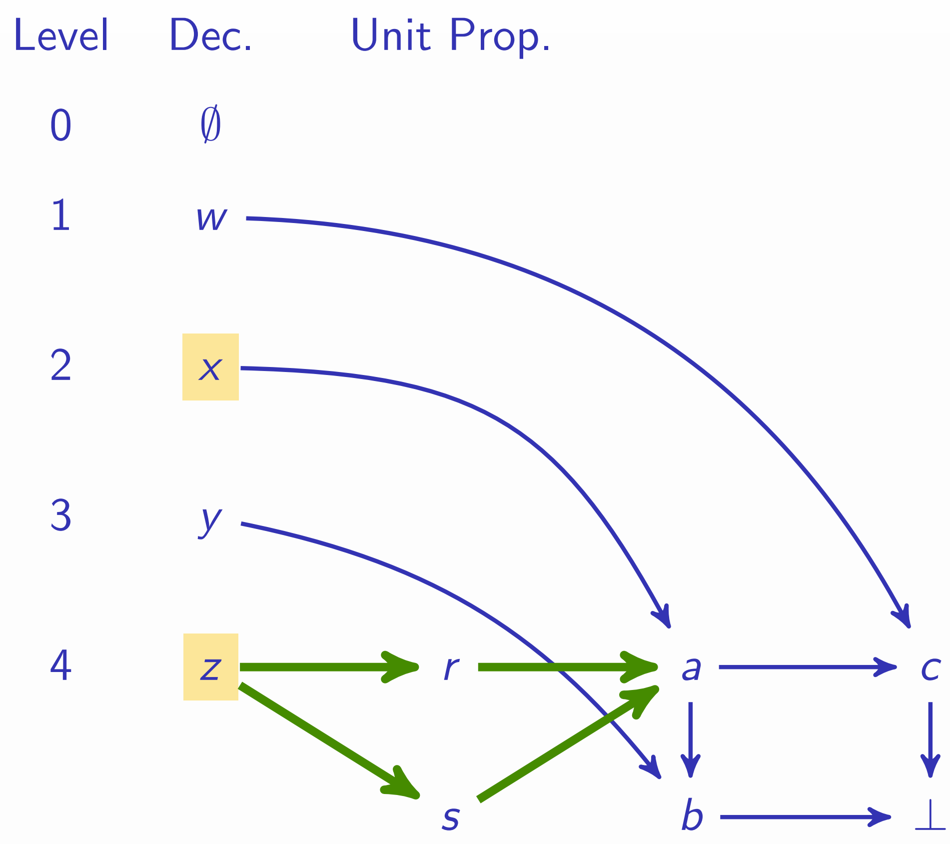

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- In practice smaller clauses more effective
  - Compare with $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

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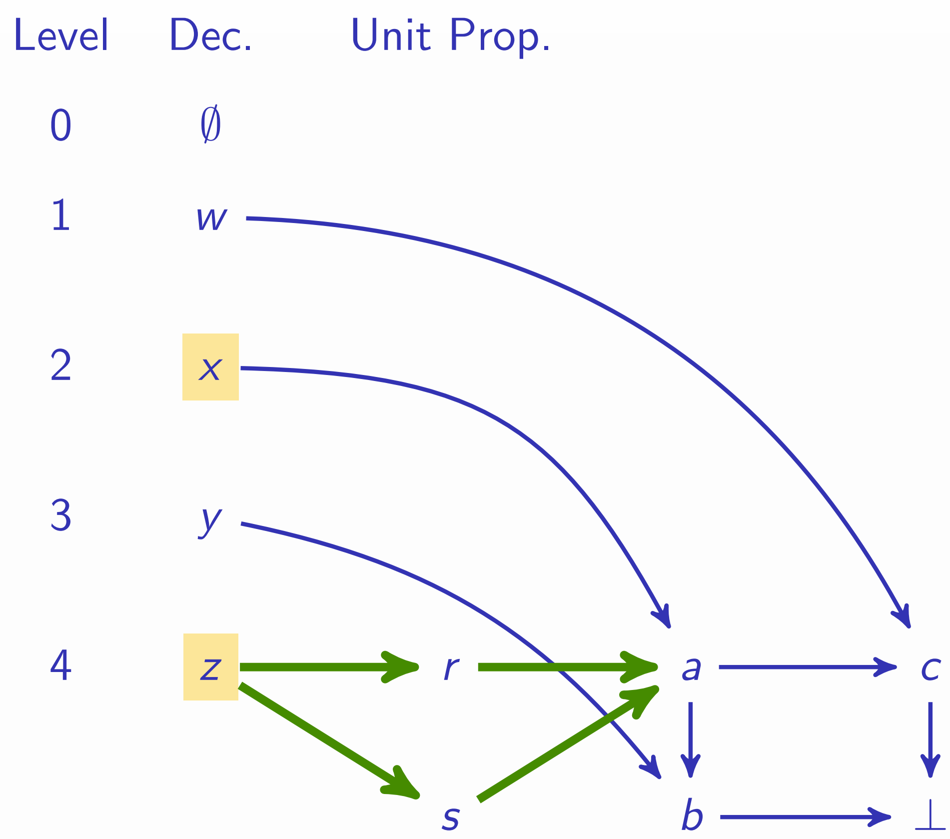

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- Multiple UIPs proposed in GRASP [MSS96]
  - First UIP learning proposed in Chaff [MMZZM01]
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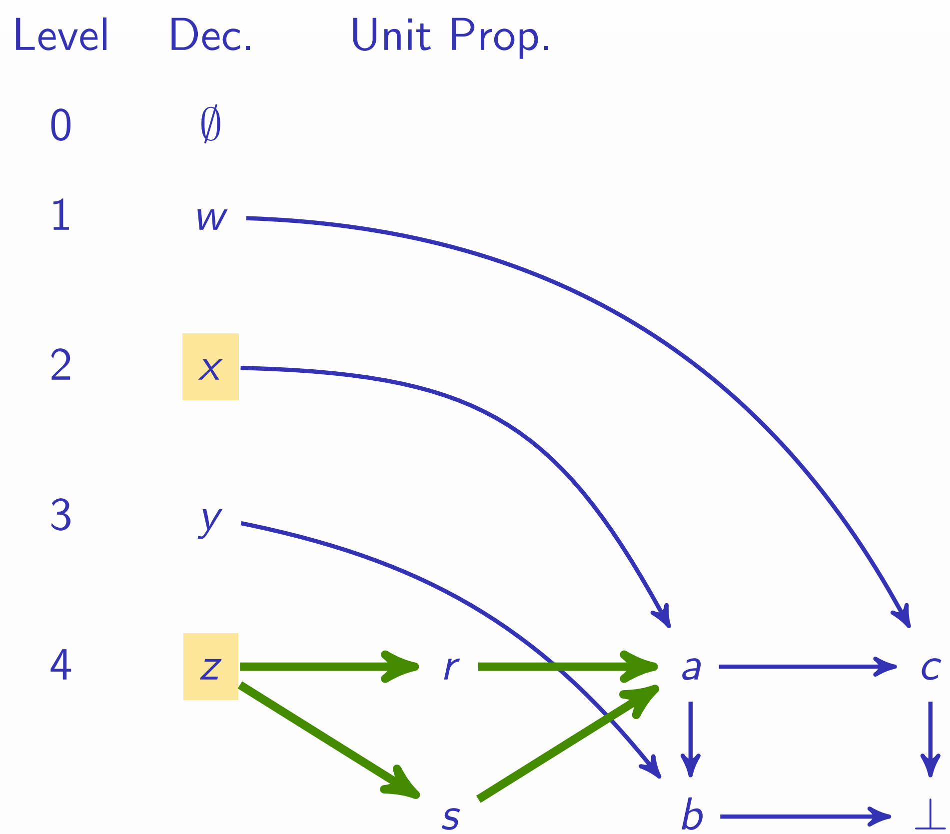
Multiple UIPs



- First UIP:
  - Learn clause $(\bar{w} \vee \bar{y} \vee \bar{a})$
- But there can be more than 1 UIP
- Second UIP:
  - Learn clause $(\bar{x} \vee \bar{z} \vee a)$
- In practice smaller clauses more effective
  - Compare with $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

- Multiple UIPs proposed in GRASP [MSS96]
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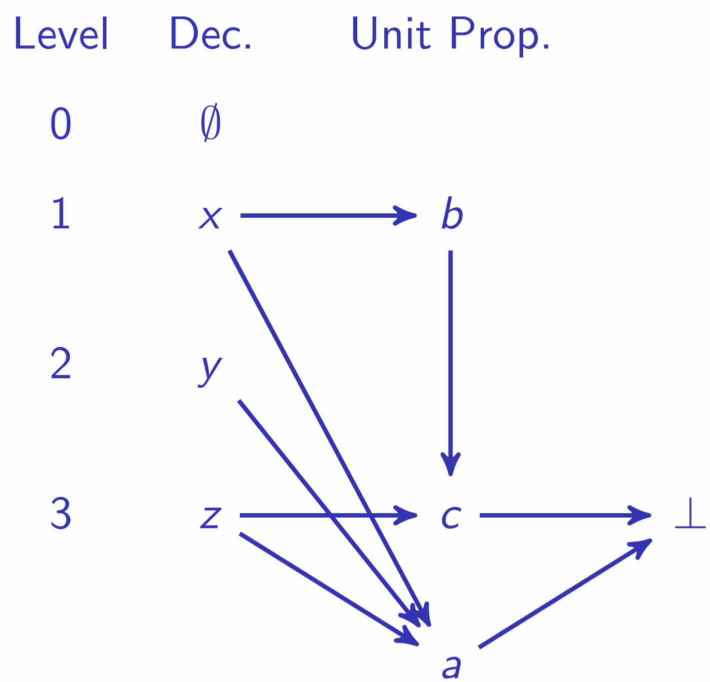
- Recent results show it can be beneficial on current instances [SSS12]

Clause Minimization I



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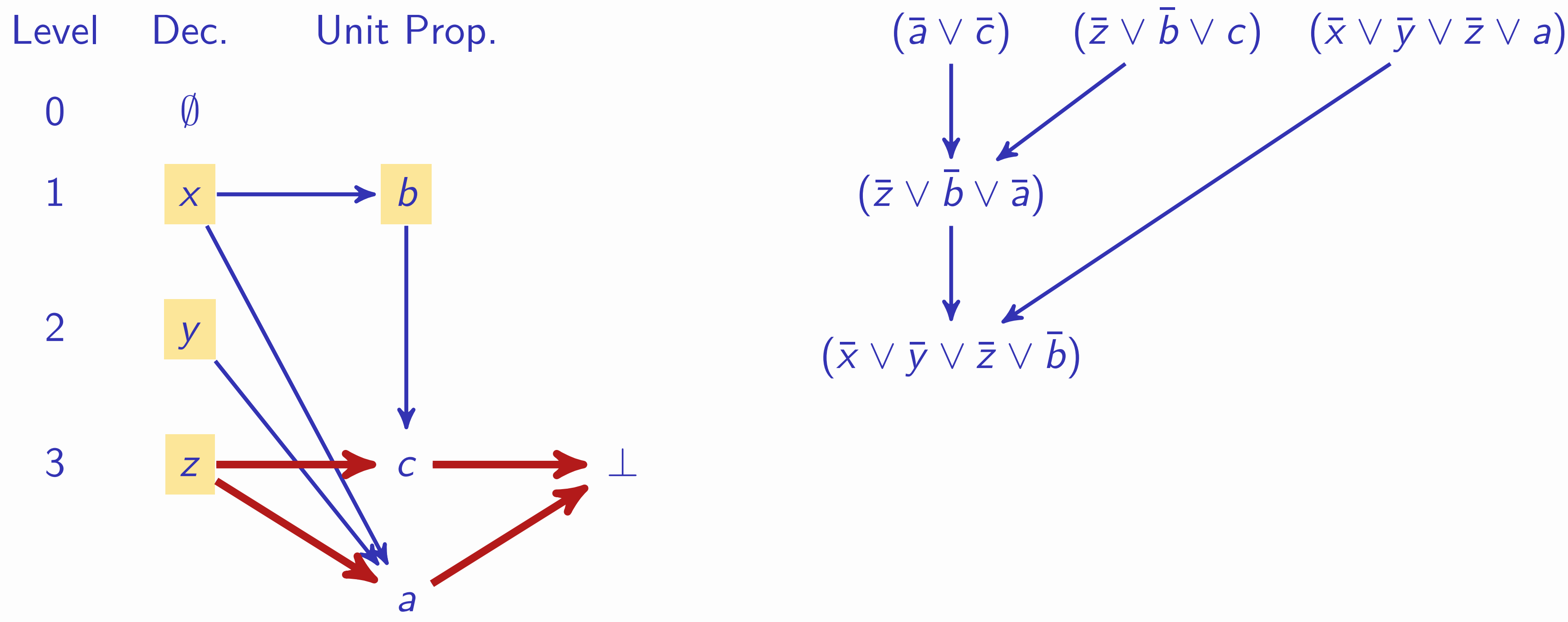

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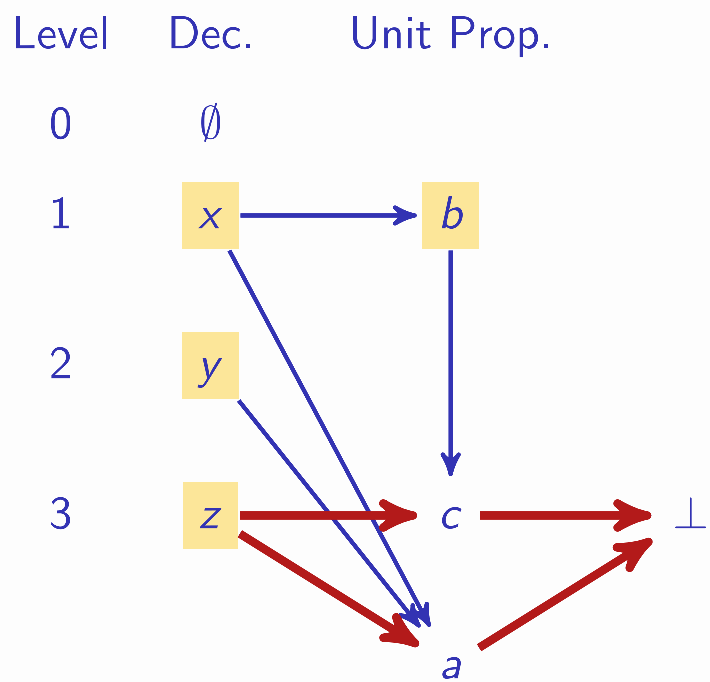

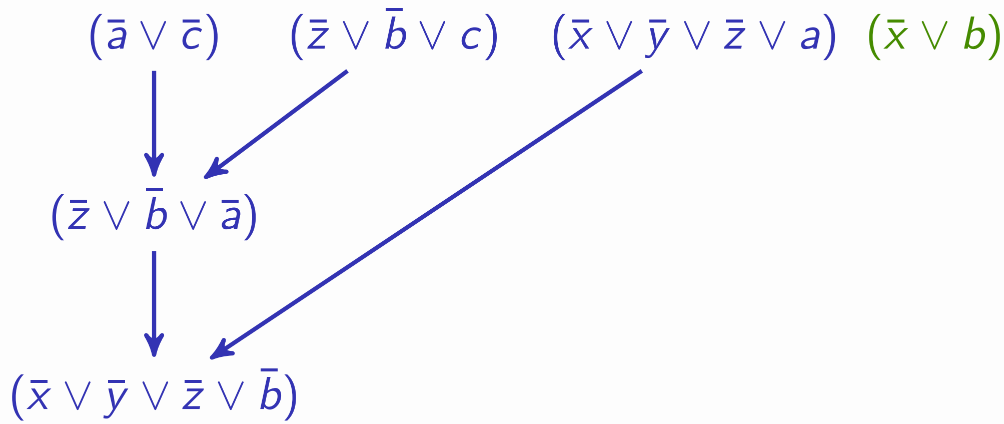

- Learn clause $(\bar{x} \lor \bar{y} \lor \bar{z} \lor \bar{b})$
- Apply self-subsuming resolution (i.e. local minimization) [SB09]

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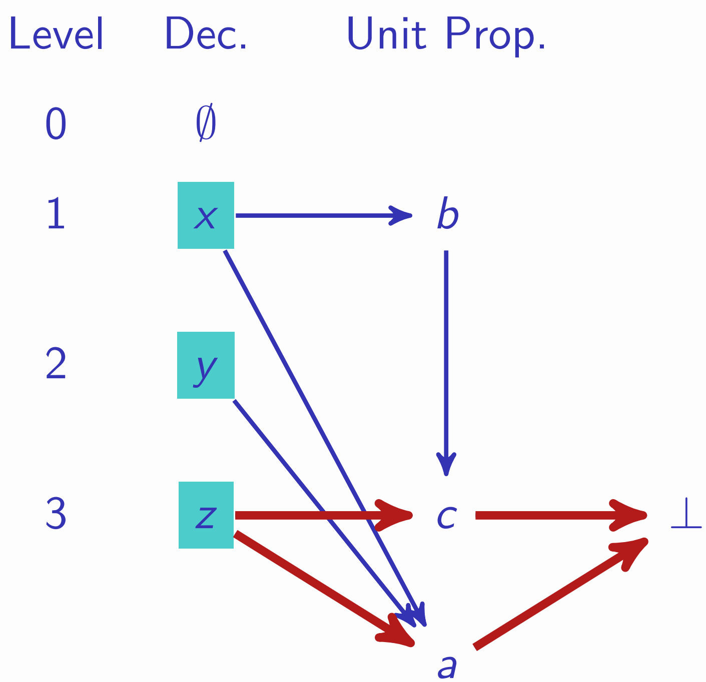

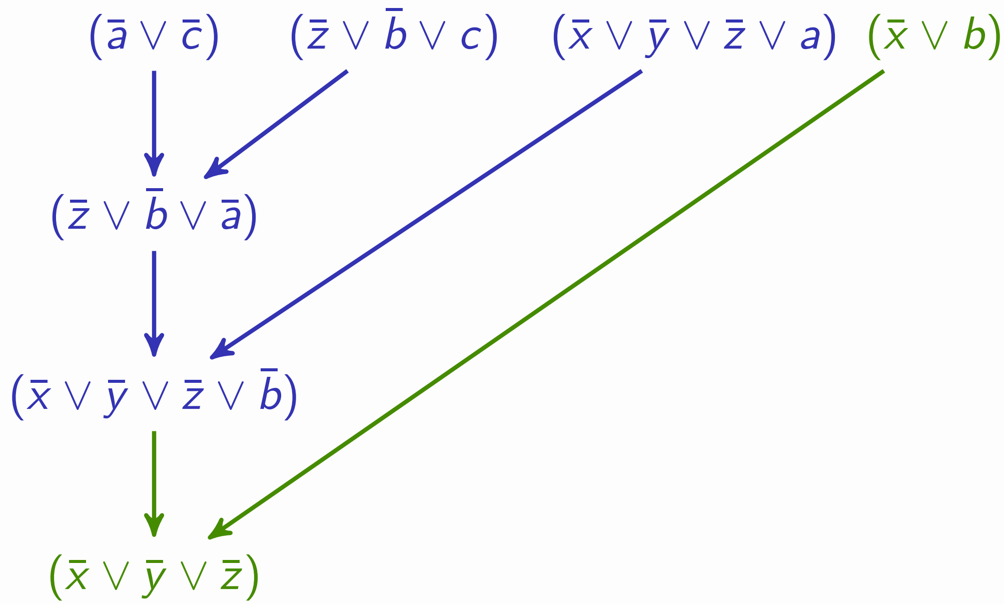

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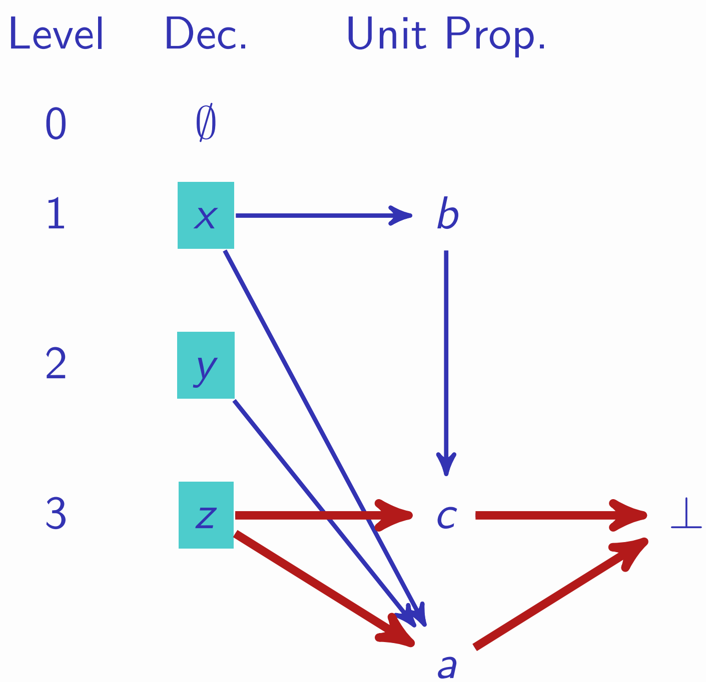

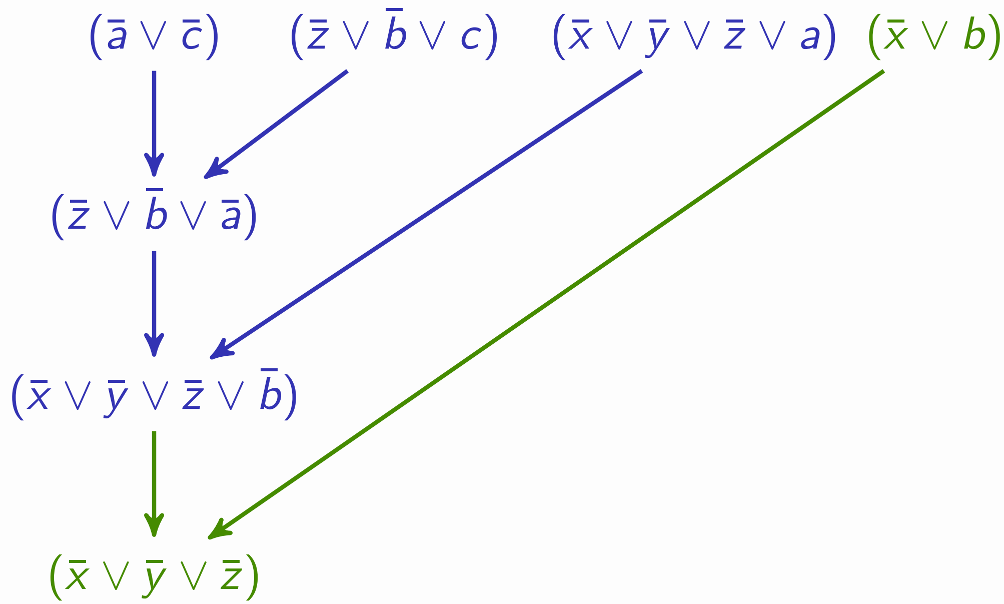

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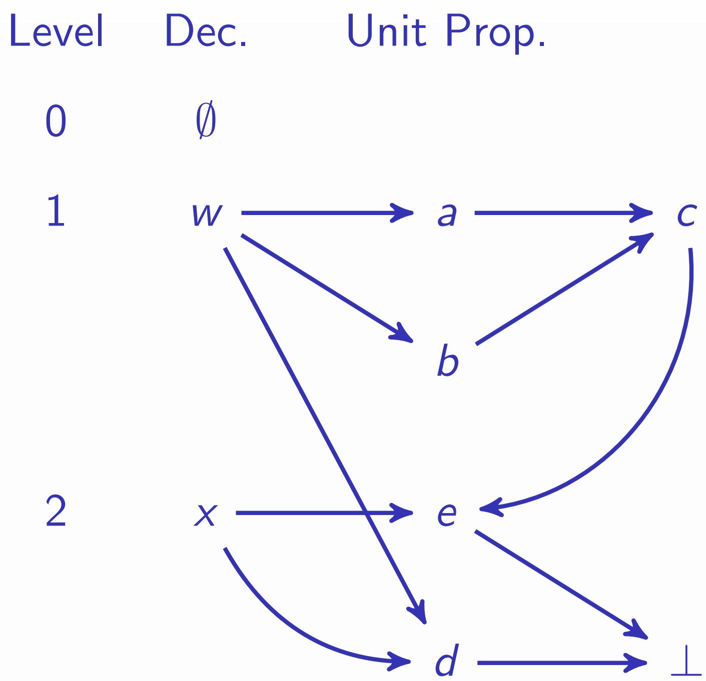

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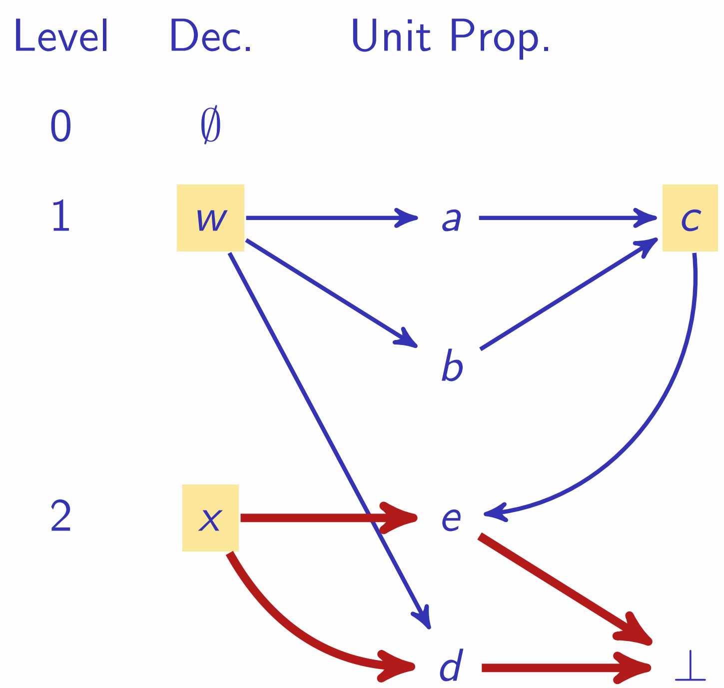

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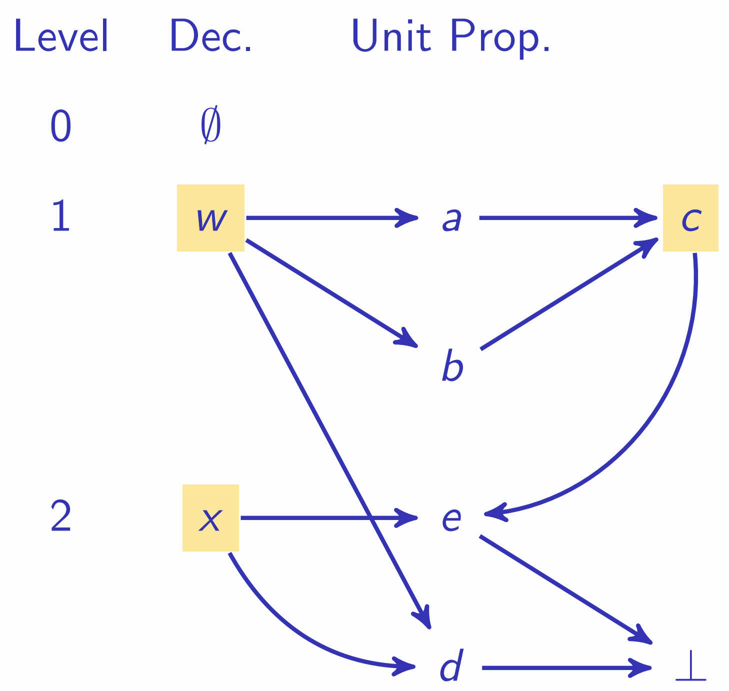

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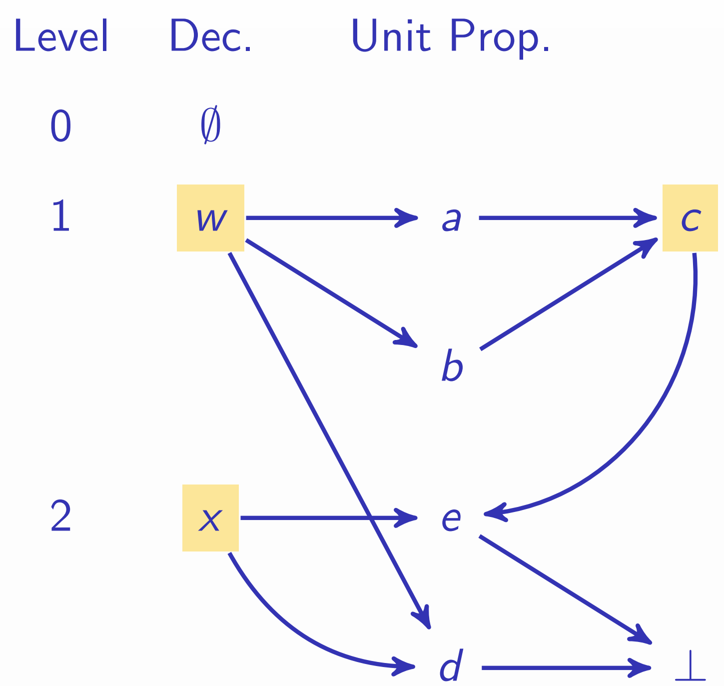

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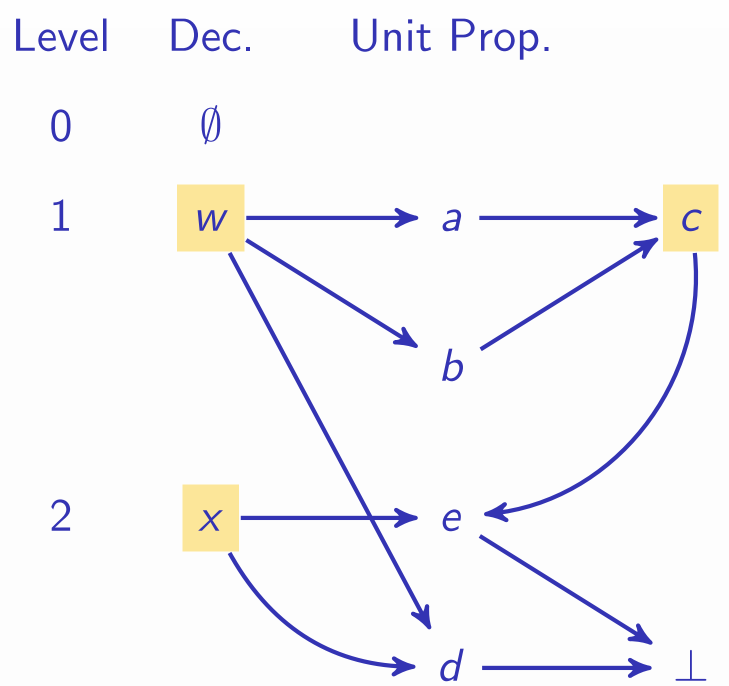

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[SB09]

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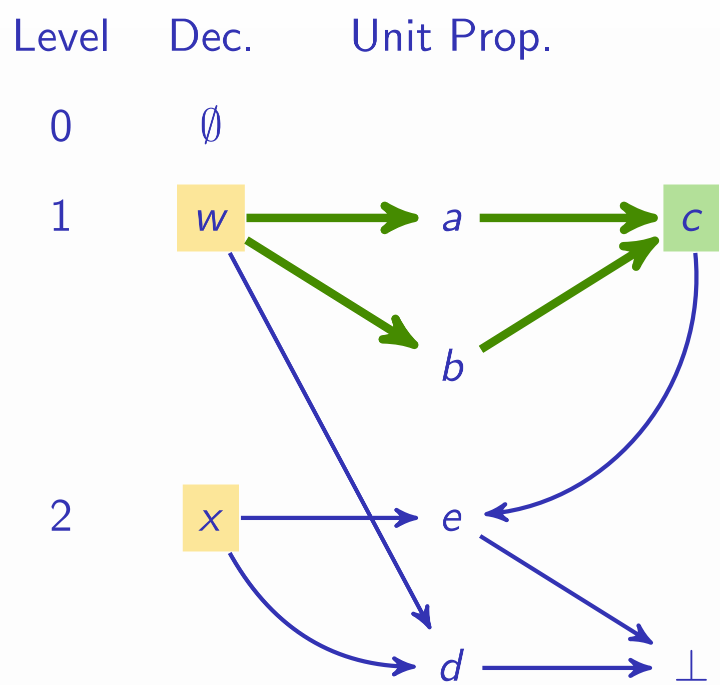

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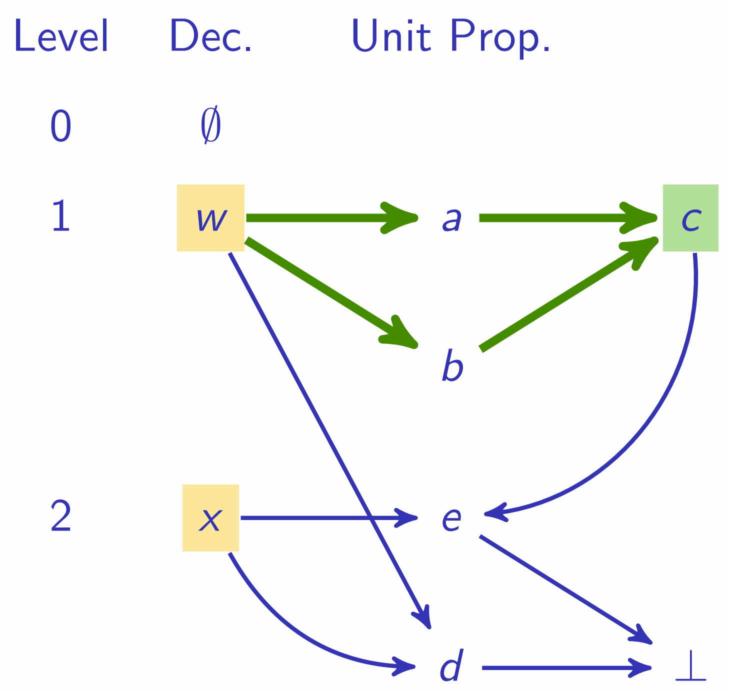

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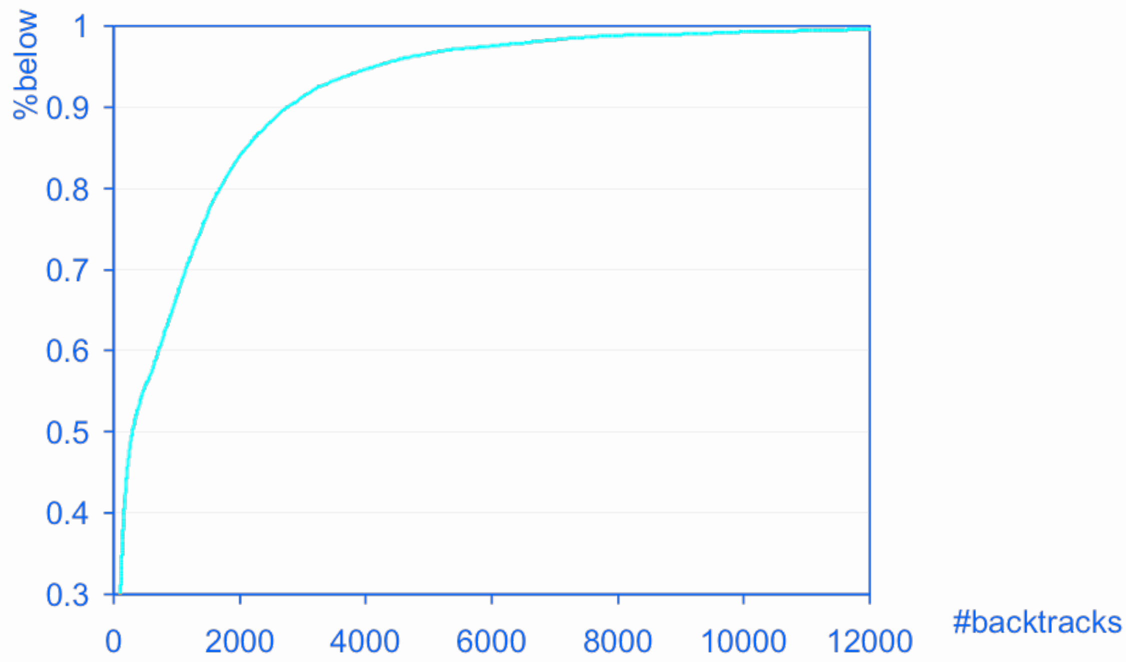
Outline

Search Restarts I

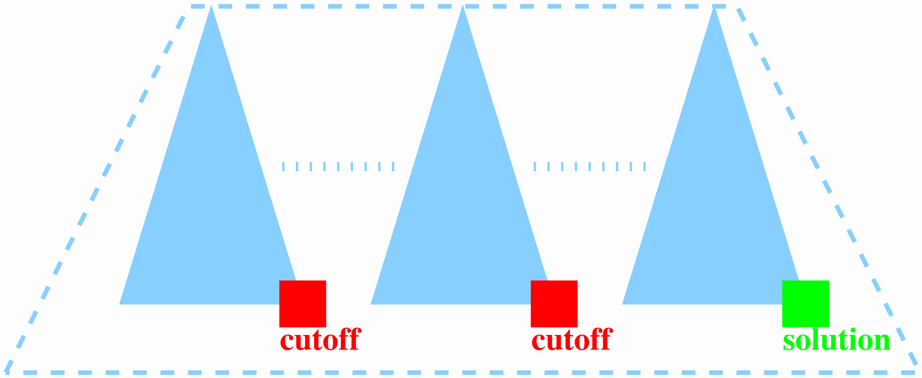
- Heavy-tail behavior: [GSK98]



  – 10000 runs, branching randomization on industrial instance
    - Use rapid randomized restarts (search restarts)

Search Restarts II

- Restart search after a number of conflicts



Search Restarts II

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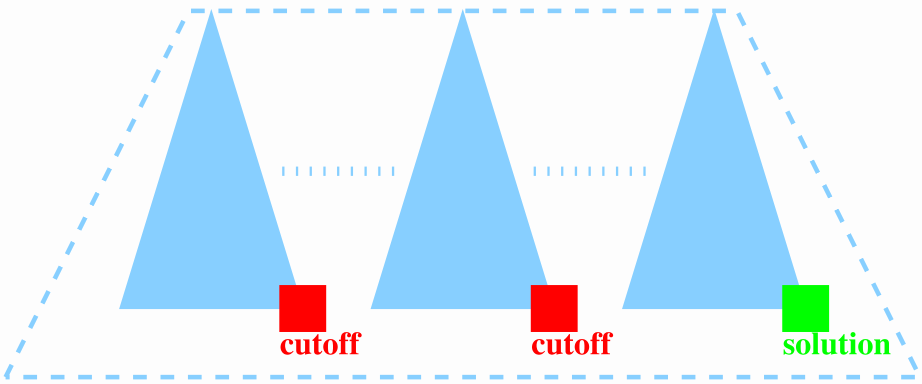
- Increase cutoff after each restart
  - Guarantees completeness
  - Different policies exist (see refs)



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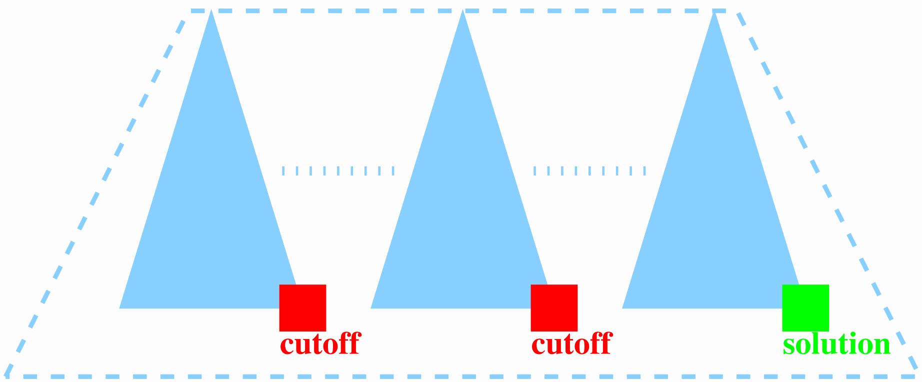

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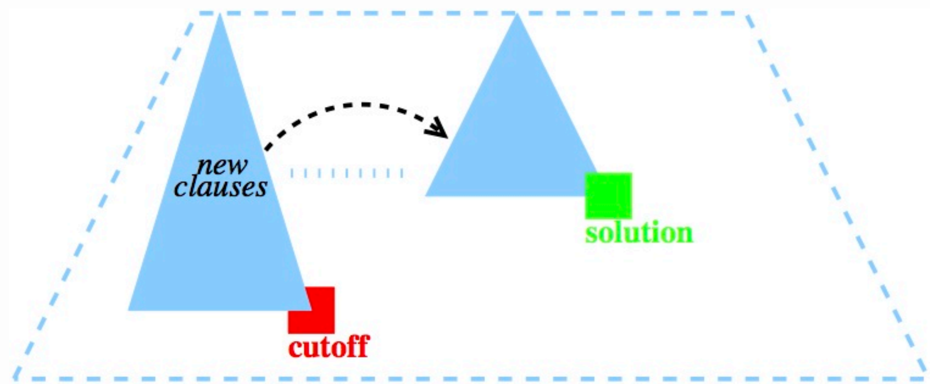

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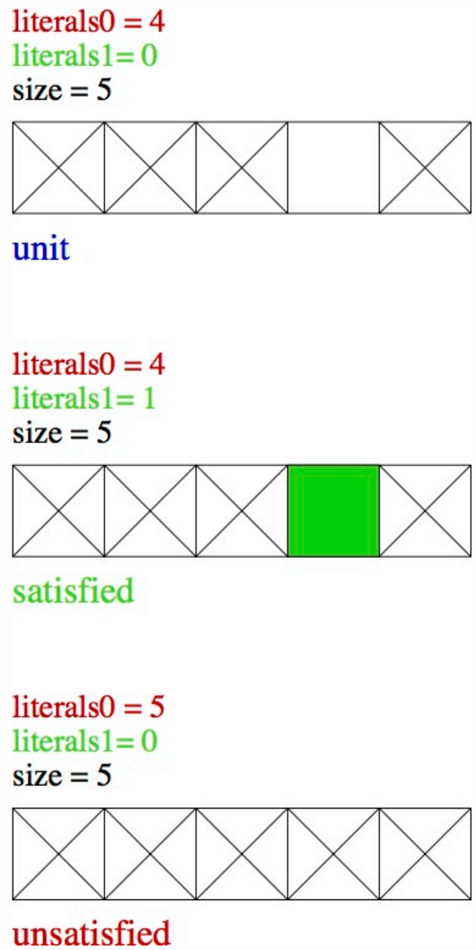
- Clause learning to be effective requires a more efficient representation: **Watched Literals**
  - Watched literals are one example of lazy data structures
    - But there are others

Watched Literals

[MMZZM01]

- Important states of a clause



Watched Literals

[MMZZM01]

- Important states of a clause
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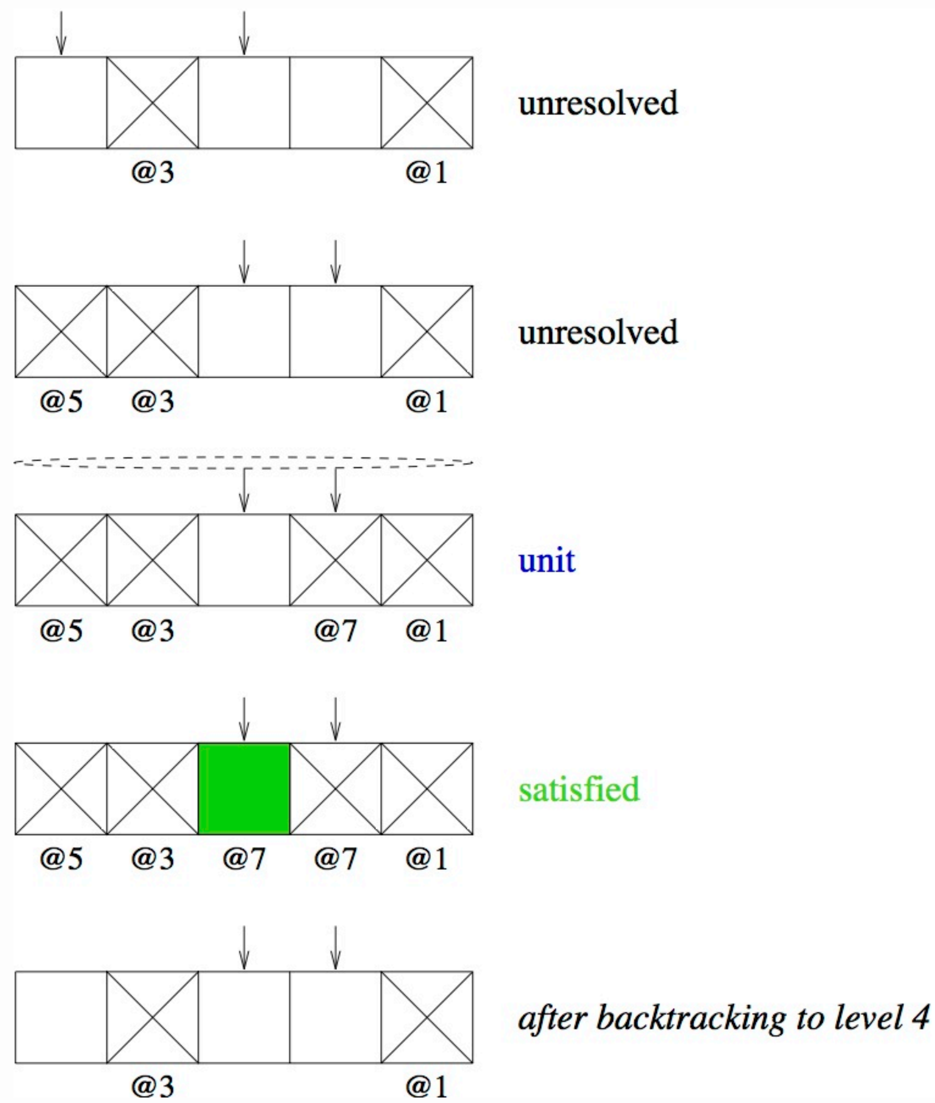

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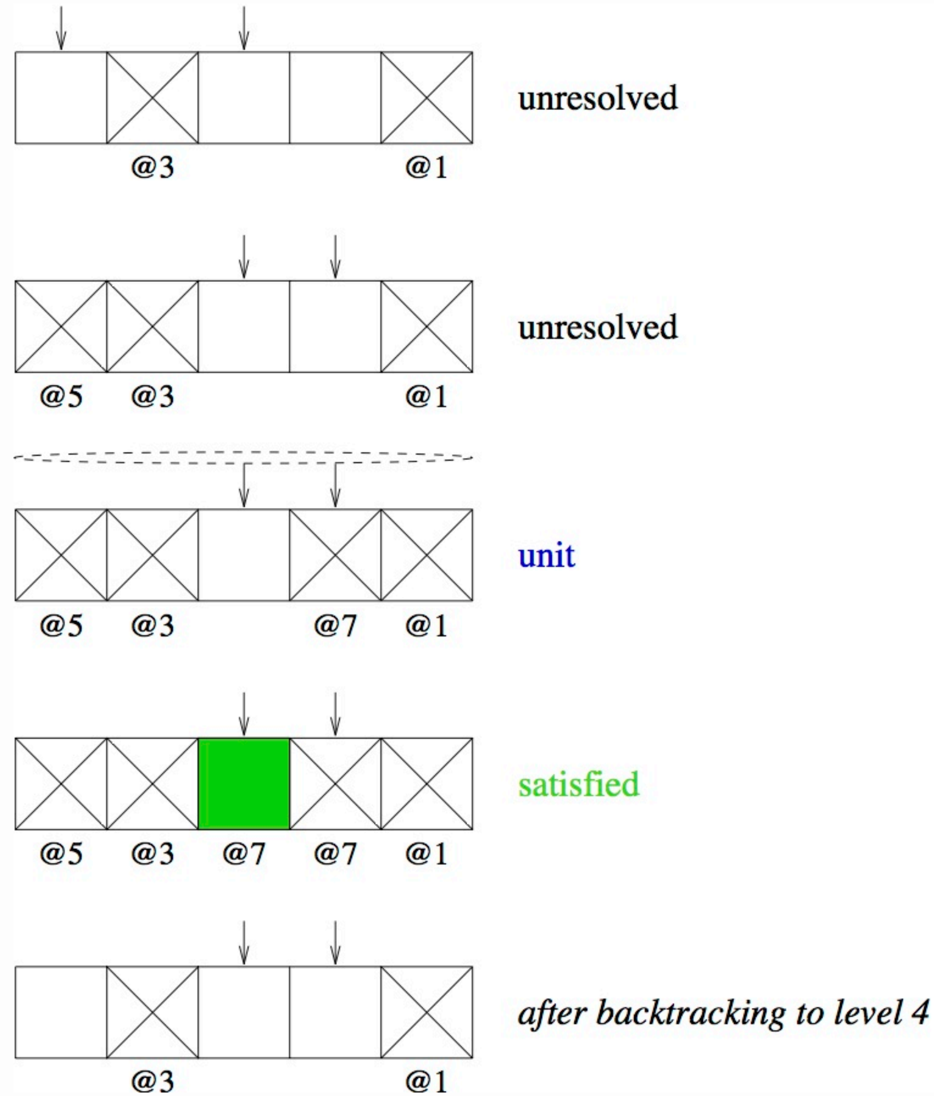

Watched Literals

[MMZZM01]

- Important states of a clause
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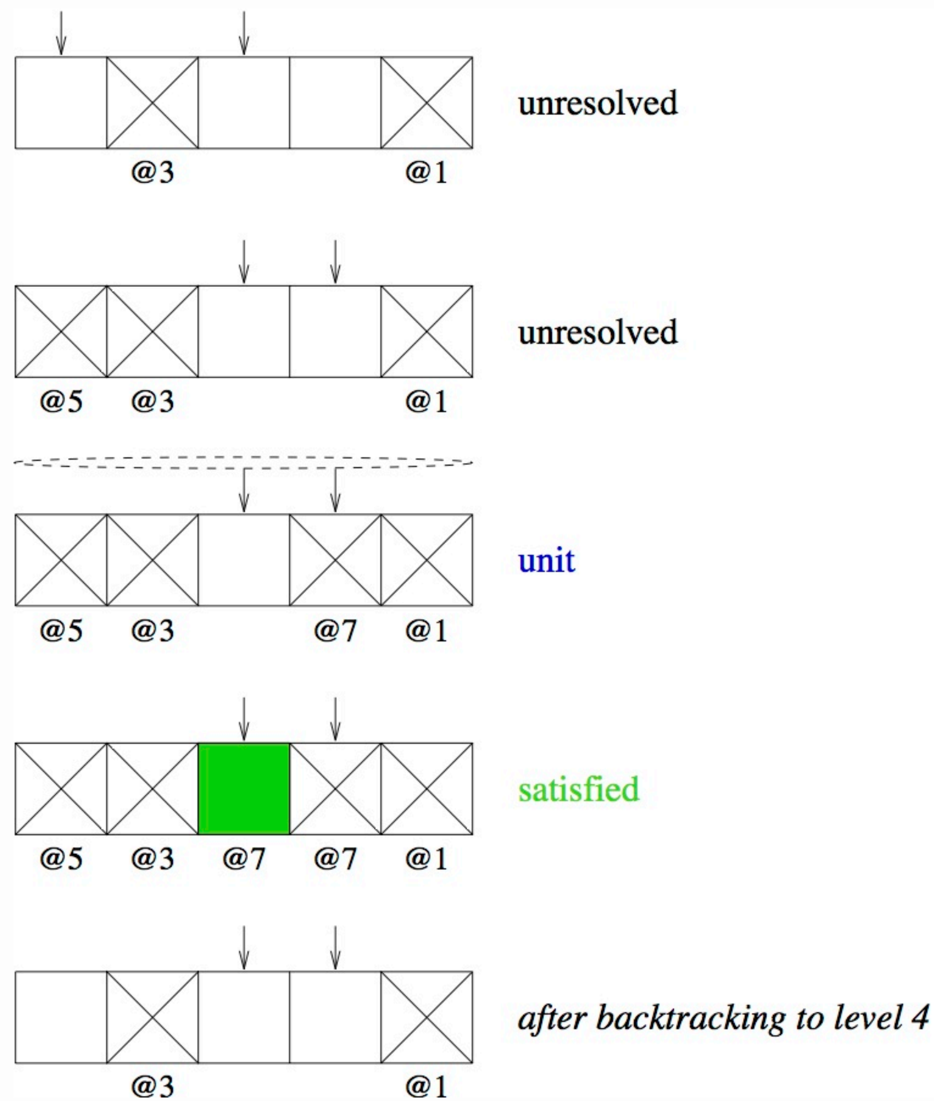
- Deciding unit requires traversing all literals
- References **unchanged** when backtracking



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  - Delete less used clauses [e.g. MSS96,GN02,ES03]
- Proven recent techniques:
  - Phase saving [PD07]
  - Literal blocks distance [AS09]

Outline

CDCL – A Glimpse of the Future

- Clause learning techniques [e.g. ABHJS08,AS09]
  - Clause learning is the key technique in CDCL SAT solvers
  - Many recent papers propose improvements to the basic clause learning approach
- Preprocessing & inprocessing
  - Many recent papers [e.g. JHB12,HJB11]
  - Essential in some applications
- Application-driven improvements
  - Incremental SAT
    - Handling of assumptions due to MUS extractors [LB13]

Part II

SAT-Based Problem Solving

How to Solve Problems with SAT?

- CNF encodings
  - Represent problem as instance of SAT
  - E.g. Eager SMT, Pseudo-Boolean constraints, etc.

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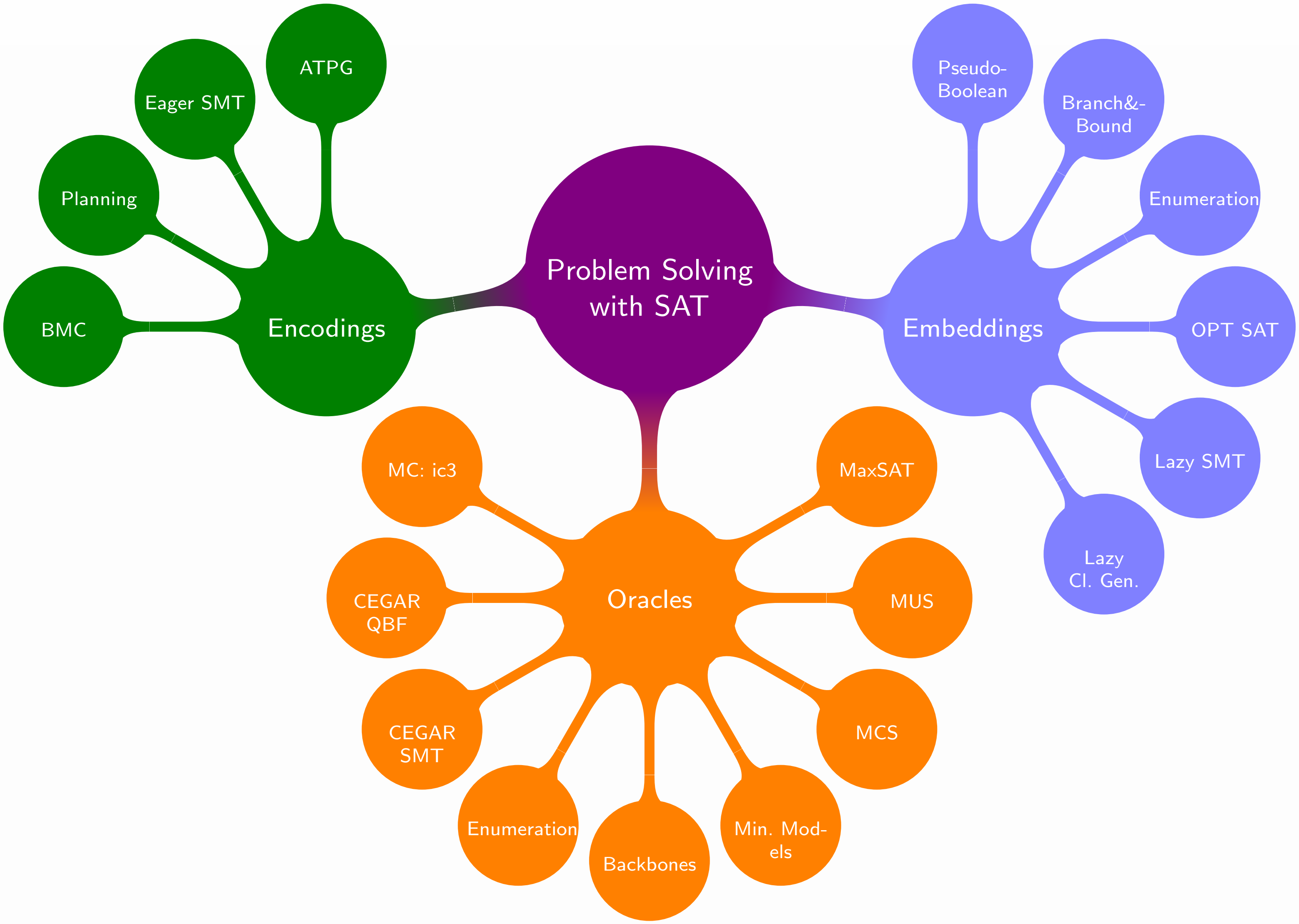
- **Note**:
  - CNF encodings most often used with either black-box or white-box approaches
  - SAT techniques adapted in many other domains: QBF, ASP, ILP, CSP, ...

SAT-Based Problem Solving



- Some apps associated with more than one concept: planning, BMC, lazy clause generation, etc.

Examples of SAT-Based Problem Solving I

- Function problems in $\mathrm{FP}^{\mathrm{NP}}[\log n]$
  - Unweighted Maximum Satisfiability (MaxSAT)
  - Minimal Correction Subsets (MCSes)
  - Minimal models
  - ...
- Function problems in $\mathrm{FP}^{\mathrm{NP}}$
  - Weighted Maximum Satisfiability (MaxSAT)
  - Minimal Unsatisfiable Subformulas (MUSes)
  - Minimal Equivalent Subformulas (MESes)
  - Prime implicates
  - ...
- Enumeration problems
  - Models
  - MUSes
  - MCSes
  - MaxSAT
  - ...

Examples of SAT-Based Problem Solving II

- Decision problems in Σ_2^P
  - 2QBF
  - ...
- Function problems in $FP^{\Sigma_2^P}$
  - (Weighted) Quantified MaxSAT (QMaxSAT) [IJMS13]
  - Smallest MUS (SMUS) [IJMS13]
  - ...
- Decision problems in PSPACE
  - QBF
  - ...
- ...

Outline

CNF Encodings

SAT Embeddings

SAT Oracles

What Next in SAT-Based Problem Solving?

Outline

CNF Encodings

SAT Embeddings

SAT Oracles

What Next in SAT-Based Problem Solving?

Encoding to CNF

- What to encode?
  - Boolean formulas
    - Tseitin's encoding
    - Plaisted&Greenbaum's encoding
    - ...
  - Cardinality constraints
  - Pseudo-Boolean (PB) constraints
  - Can also translate to SAT:
    - Constraint Satisfaction Problems (CSPs)
    - Answer Set Programming (ASP)
    - Model Finding
    - ...

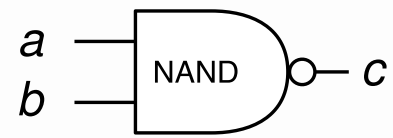
- Key issues:
  - Encoding size
  - Arc-consistency?

Outline

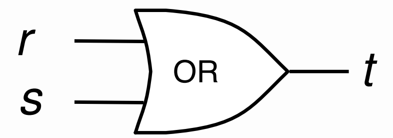
Representing Boolean Formulas / Circuits I

- Satisfiability problems can be defined on Boolean circuits/formulas
- Can represent circuits/formulas as CNF formulas [T68,PG86]
  - For each (simple) gate, CNF formula encodes the consistent assignments to the gate's inputs and output
    - Given $z = \mathrm{OP}(x, y)$, represent in CNF $z \leftrightarrow \mathrm{OP}(x, y)$
  - CNF formula for the circuit is the conjunction of CNF formula for each gate

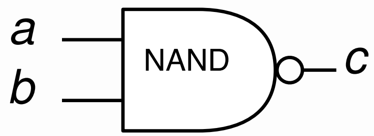
$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$



$$\mathcal{F}_t = (\bar{r} \vee t) \wedge (\bar{s} \vee t) \wedge (r \vee s \vee \bar{t})$$



Representing Boolean Formulas / Circuits II

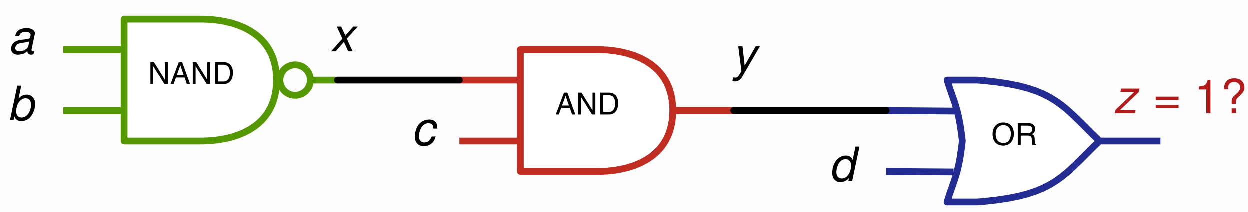


a	b	c	$\mathcal{F}_c$(a,b,c)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$

Representing Boolean Formulas / Circuits III

- CNF formula for the circuit is the conjunction of the CNF formula for each gate
  - Can specify objectives with additional clauses



$$
\begin{aligned}
\mathcal{F} \quad = \quad & (a \vee x) \wedge (b \vee x) \wedge (\bar{a} \vee \bar{b} \vee \bar{x}) \wedge \\
& (x \vee \bar{y}) \wedge (c \vee \bar{y}) \wedge (\bar{x} \vee \bar{c} \vee y) \wedge \\
& (\bar{y} \vee z) \wedge (\bar{d} \vee z) \wedge (y \vee d \vee \bar{z}) \wedge (z)
\end{aligned}
$$

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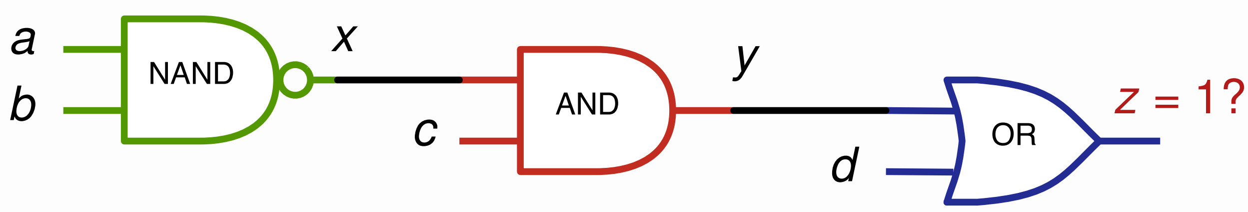

$$\begin{aligned}\mathcal{F} \;=\; & (a \vee x) \wedge (b \vee x) \wedge (\bar{a} \vee \bar{b} \vee \bar{x}) \wedge \\ & (x \vee \bar{y}) \wedge (c \vee \bar{y}) \wedge (\bar{x} \vee \bar{c} \vee y) \wedge \\ & (\bar{y} \vee z) \wedge (\bar{d} \vee z) \wedge (y \vee d \vee \bar{z}) \wedge (z)\end{aligned}$$

- Note: $z = d \vee (c \wedge (\neg(a \wedge b)))$
  - **No** distinction between Boolean circuits and formulas

Outline

Cardinality Constraints

- How to handle cardinality constraints, $\sum_{j=1}^{n} x_j \leq k$?
  - How to handle AtMost1 constraints, $\sum_{j=1}^{n} x_j \leq 1$?
  - General form: $\sum_{j=1}^{n} x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$

- Solution #1:
  - Use PB solver
  - Difficult to keep up with advances in SAT technology
  - For SAT/UNSAT, best solvers already encode to CNF
    - E.g. Minisat+, but also QMaxSat, MSUnCore, (W)PM2

Cardinality Constraints

- How to handle cardinality constraints, $\sum_{j=1}^{n} x_j \leq k$?
  - How to handle AtMost1 constraints, $\sum_{j=1}^{n} x_j \leq 1$?
  - General form: $\sum_{j=1}^{n} x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$
- Solution #1:
  - Use PB solver
  - Difficult to keep up with advances in SAT technology
  - For SAT/UNSAT, best solvers already encode to CNF
    - E.g. Minisat+, but also QMaxSat, MSUnCore, (W)PM2
- Solution #2:
  - Encode cardinality constraints to CNF
  - Use SAT solver

Equals1, AtLeast1 & AtMost1 Constraints

- $\sum_{j=1}^{n} x_j = 1$: encode with $(\sum_{j=1}^{n} x_j \leq 1) \wedge (\sum_{j=1}^{n} x_j \geq 1)$
- $\sum_{j=1}^{n} x_j \geq 1$: encode with $(x_1 \vee x_2 \vee \ldots \vee x_n)$
- $\sum_{j=1}^{n} x_j \leq 1$ encode with:
  - Pairwise encoding
    - Clauses: $\mathcal{O}(n^2)$; No auxiliary variables
  - Sequential counter [S05]
    - Clauses: $\mathcal{O}(n)$; Auxiliary variables: $\mathcal{O}(n)$
  - Bitwise encoding [P07,FP01]
    - Clauses: $\mathcal{O}(n \log n)$; Auxiliary variables: $\mathcal{O}(\log n)$
  - ...

Bitwise Encoding

- Encode $\sum_{j=1}^{n} x_j \leq 1$ with bitwise encoding:

- An example: $x_1 + x_2 + x_3 \leq 1$

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  - Auxiliary variables $v_0, \ldots, v_{r-1}$; $r = \lceil \log n \rceil$ (with $n > 1$)
  - If $x_j = 1$, then $v_0 \ldots v_{r-1} = b_0 \ldots b_{r-1}$, the binary encoding of $j - 1$

    $x_j \rightarrow (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}) \Leftrightarrow (\bar{x}_j \vee (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}))$

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	$j-1$	$v_1 v_0$
x_1	0	00
x_2	1	01
x_3	2	10

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  - Clauses $(\bar{x}_j \vee (v_i \leftrightarrow b_i)) = (\bar{x}_j \vee l_i)$, $i = 0, \ldots, r-1$, where
    - $l_i \equiv v_i$, if $b_i = 1$
    - $l_i \equiv \bar{v}_i$, otherwise

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$$(\bar{x}_1 \vee \bar{v}_1) \wedge (\bar{x}_1 \vee \bar{v}_0)$$
$$(\bar{x}_2 \vee \bar{v}_1) \wedge (\bar{x}_2 \vee v_0)$$
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    - $l_i \equiv v_i$, if $b_i = 1$
    - $l_i \equiv \bar{v}_i$, otherwise
  - If $x_j = 1$, assignment to v_i variables must encode $j - 1$
    - All other x variables must take value 0
  - If all $x_j = 0$, any assignment to v_i variables is consistent
  - $\mathcal{O}(n \log n)$ clauses ; $\mathcal{O}(\log n)$ auxiliary variables
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$$
\begin{gathered}
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(\bar{x}_3 \vee v_1) \wedge (\bar{x}_3 \vee \bar{v}_0)
\end{gathered}
$$

General Cardinality Constraints

- General form: $\sum_{j=1}^{n} x_j \leq k$ (or $\sum_{j=1}^{n} x_j \geq k$)
  - Sequential counters [S05]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - BDDs [ES06]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - Sorting networks [ES06]
    - Clauses/Variables: $\mathcal{O}(n \log^2 n)$
  - Cardinality Networks: [ANORC09,ANORC11a]
    - Clauses/Variables: $\mathcal{O}(n \log^2 k)$
  - Pairwise Cardinality Networks: [CZI10]
  - ...

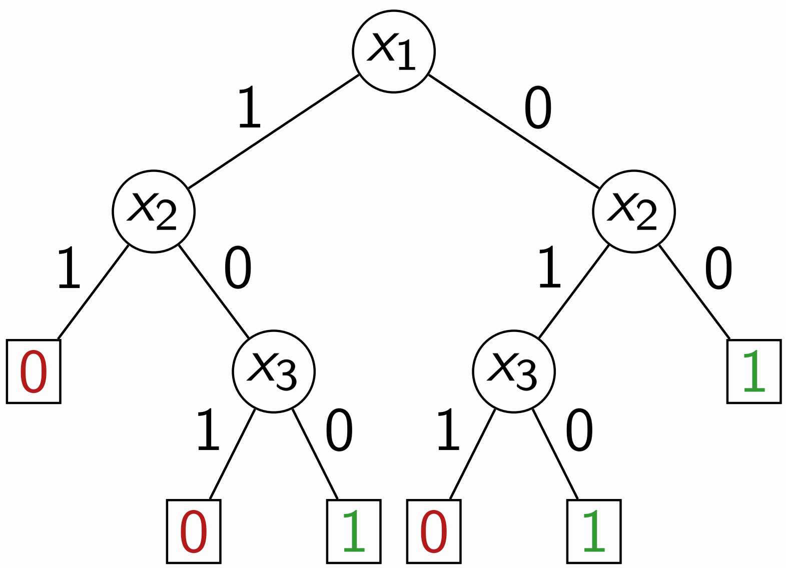
Outline

Pseudo-Boolean Constraints

- General form: $\sum_{j=1}^{n} a_j x_j \leq b$
  - Operational encoding [W98]
    - Clauses/Variables: $\mathcal{O}(n)$
    - Does not guarantee arc-consistency
  - BDDs [ES06]
    - Worst-case exponential number of clauses
  - Polynomial watchdog encoding [BBR09]
    - Let $\nu(n) = \log(n)\log(a_{max})$
    - Clauses: $\mathcal{O}(n^3\nu(n))$; Aux variables: $\mathcal{O}(n^2\nu(n))$
  - Improved polynomial watchdog encoding [ANORC11b]
    - Clauses & aux variables: $\mathcal{O}(n^3 \log(a_{max}))$
  - ...

Encoding PB Constraints with BDDs I

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Construct BDD
  - E.g. analyze variables by decreasing coefficients
- Extract ITE-based circuit from BDD



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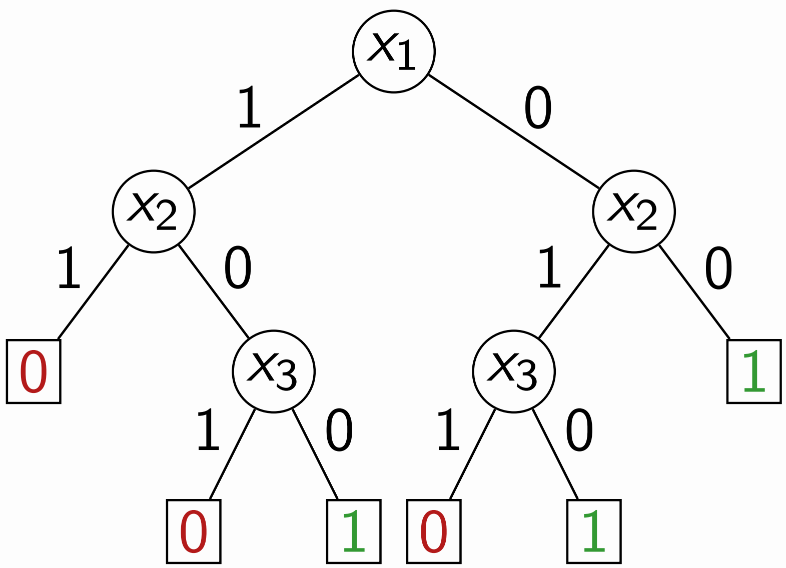
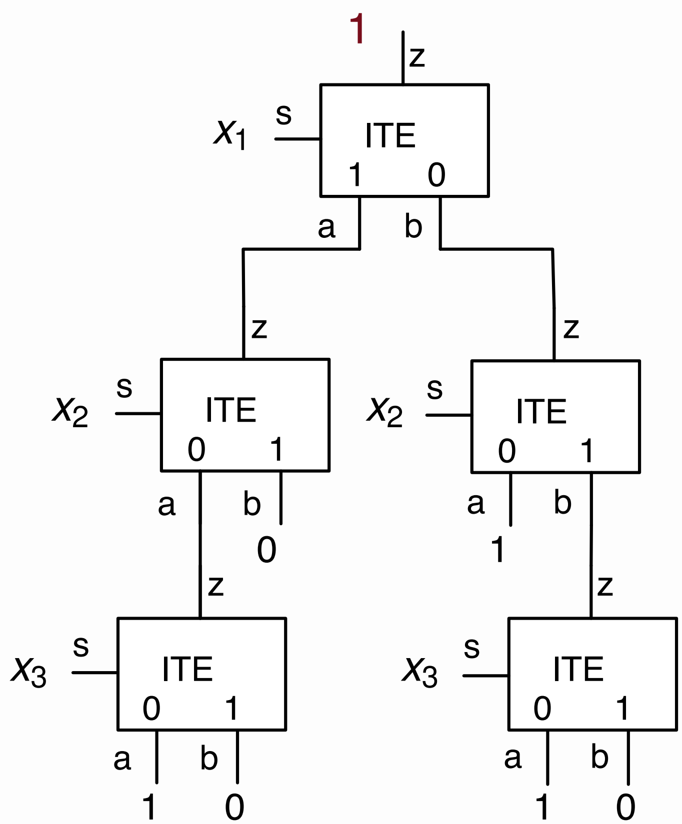
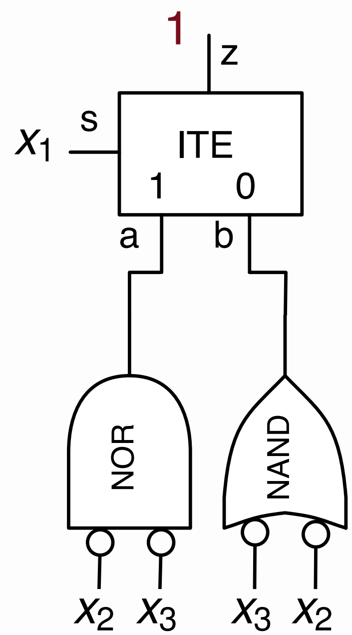




Encoding PB Constraints with BDDs II

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Extract ITE-based circuit from BDD
- Simplify and create final circuit:



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  - Let $x_2 = 0$
  - Either constraint can still be satisfied, but **not** both

Outline

CSP Constraints

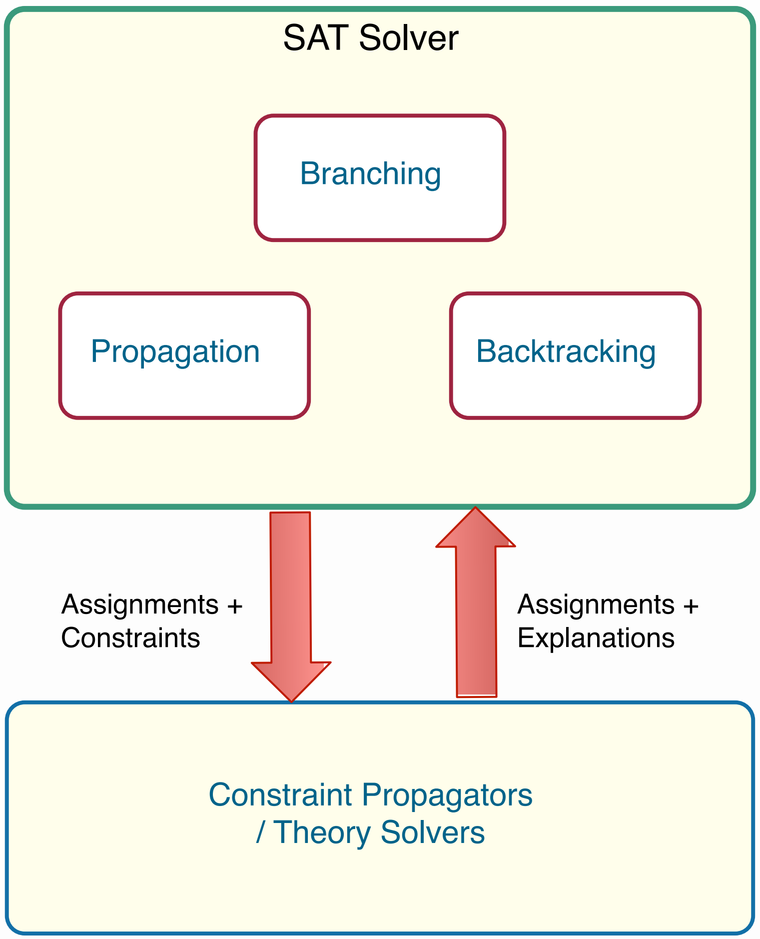
- Many possible encodings:
  - Direct encoding [dK89,GJ96,W00]
  - Log encoding [W00]
  - Support encoding [K90,G02]
  - Log-Support encoding [G07]
  - Order encoding for finite linear CSPs [TTKB09]

Direct Encoding for CSP w/ Binary Constraints

- Variable x_i with domain D_i, with $m_i = |D_i|$
- Represent values of x_i with Boolean variables $x_{i,1}, \ldots, x_{i,m_i}$
- Require $\sum_{k=1}^{m_i} x_{i,k} = 1$
  - Suffices to require $\sum_{k=1}^{m_i} x_{i,k} \geq 1$ [W00]
- If the pair of assignments $x_i = v_i \wedge x_j = v_j$ is not allowed, add binary clause $(\bar{x}_{i,v_i} \vee \bar{x}_{j,v_j})$

Outline

Embedding SAT Solvers



- Modify SAT solver to interface problem-specific propagators (or theory solvers)
- Typical interface:
  - SAT solvers communicates assignments/constraints to propagators
  - Retrieve resulting assignments or explanations for inconsistency
- Well-known examples (many more):
  - Branch&bound PB optimization
  - Non-clausal SAT solvers
  - Lazy SMT solving (see later talks)
- Key problem:
  - Keeping up with improvements in SAT solvers

Pseudo-Boolean Constraints & Optimization

- Pseudo-Boolean Constraints:
  - Boolean variables: $x_1, \ldots, x_n$
  - Linear inequalities:

$$\sum_{j \in N} a_{ij} l_j \geq b_i, \quad l_j \in \{x_j, \bar{x}_j\}, x_j \in \{0, 1\}, a_{ij}, b_i \in \mathbb{N}_0^+$$

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- Pseudo-Boolean Optimization (PBO):

$$\begin{array}{ll} \text{minimize} & \sum_{j \in N} c_j \cdot x_j \\ \text{subject to} & \sum_{j \in N} a_{ij} l_j \geq b_i, \\ & l_j \in \{x_j, \bar{x}_j\}, x_j \in \{0, 1\}, a_{ij}, b_i, c_j \in \mathbb{N}_0^+ \end{array}$$

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- Branch and bound (B&B) PBO algorithm: [MMS00]
  - Extend SAT solver
  - Must develop propagator for PB constraints
  - B&B search for computing optimum cost function value
    - Trivial upper bound: all $x_j = 1$

Limitations with Embeddings

- B&B MaxSAT solving:
  - Cannot use unit propagation
  - Cannot learn clauses

- MUS extraction:
  - Decision of clauses to include in MUS based on unsatisfiable outcomes
  - No immediate gain from embedding SAT solvers

Outline

Practical Aspects of Using SAT Oracles

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  4. Compute proof traces/resolution proof: $(st, \mu, \mathcal{T}) \leftarrow \text{SAT}(\mathcal{F})$

Outline

CNF Encodings

SAT Embeddings

SAT Oracles
- MUS Extraction
- MaxSAT
- 2QBF

What Next in SAT-Based Problem Solving?

Defining MUSes

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Formula is unsatisfiable but not irreducible

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$$
\begin{array}{cccc}
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\neg x_6 \vee x_8 & x_6 \vee \neg x_8 & x_2 \vee x_4 & \neg x_4 \vee x_5 \\
\boxed{x_7 \vee x_5 \quad \neg x_7 \vee x_5 \quad \neg x_5 \vee x_3 \quad \neg x_3}
\end{array}
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$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Formula is unsatisfiable but not irreducible
- Can remove clauses, and formula still unsatisfiable
- A Minimal Unsatisfiable Subformula (**MUS**) is an unsatisfiable and irreducible subformula
- How to compute an MUS?

Deletion-Based MUS Extraction

Input : Unsatisfiable CNF Formula $\mathcal{F}$
Output: MUS $\mathcal{M}$
begin
    $\mathcal{M} \leftarrow \mathcal{F}$ // MUS over-approximation
    foreach $c \in \mathcal{M}$ **do**
        if not $\texttt{SAT}(\mathcal{M} \setminus \{c\})$ **then**
            $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // If $\texttt{UNSAT}(\mathcal{M} \setminus \{c\})$, then $c \notin \mathcal{M}$
    return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of calls to SAT solver: $\mathcal{O}(|\mathcal{F}|)$

Deletion-Based MUS Extraction

Input : Unsatisfiable CNF Formula $\mathcal{F}$
Output: MUS $\mathcal{M}$

```
begin
    M ← F                                   // MUS over-approximation
    foreach c ∈ M do
        if not SAT(M \ {c}) then
            M ← M \ {c}                     // Remove c from M
    return M                                // Final M is MUS
end
```

- Number of calls to SAT solver: $\mathcal{O}(|\mathcal{F}|)$

An Example

$$(\neg x_1 \vee x_2)$$
$$(\neg x_3 \vee x_2)$$
$$(x_1 \vee x_2)$$
$$(\neg x_3)$$
$$(\neg x_2)$$

UNSAT instance

An Example

$$(\neg x_1 \vee x_2)$$
$$(\neg x_3 \vee x_2)$$
$$(x_1 \vee x_2)$$
$$(\neg x_3)$$
$$(\neg x_2)$$

Hide clause $(\neg x_1 \vee x_2)$

An Example

$$
\begin{array}{l}
(\neg x_1 \vee x_2) \\
(\neg x_3 \vee x_2) \\
(x_1 \vee x_2) \\
(\neg x_3) \\
(\neg x_2)
\end{array}
$$

SAT instance $\rightarrow$ keep clause $(\neg x_1 \vee x_2)$

An Example

$(\neg x_1 \lor x_2)$
$(\neg x_3 \lor x_2)$
$(x_1 \lor x_2)$
$(\neg x_3)$
$(\neg x_2)$

Hide clause $(\neg x_3 \lor x_2)$

An Example

$$(\neg x_1 \vee x_2)$$
$$(\neg x_3 \vee x_2)$$
$$(x_1 \vee x_2)$$
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UNSAT instance $\rightarrow$ remove clause $(\neg x_3 \vee x_2)$

An Example

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$$(\neg x_3 \vee x_2)$$
$$(x_1 \vee x_2)$$
$$(\neg x_3)$$
$$(\neg x_2)$$

Hide clause $(x_1 \vee x_2)$

An Example

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An Example

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Hide clause $(\neg x_3)$

An Example

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UNSAT instance $\rightarrow$ remove clause $(\neg x_3)$

An Example

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Hide clause $(\neg x_2)$

An Example

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$(\neg x_3)$

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SAT instance $\rightarrow$ keep clause $(\neg x_2)$

An Example

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$(x_1 \vee x_2)$
$(\neg x_2)$

Computed MUS

More on MUS Extraction

Algorithm	# Oracle Calls	Reference
Insertion (Default)	$\mathcal{O}(m \times k)$	[SP88]
Deletion (Default)	$\mathcal{O}(m)$	[CD91,BDTW93]
QuickXplain	$\mathcal{O}(k \times (1 + \log \frac{m}{k}))$	[J01,J04]
Dichotomic	$\mathcal{O}(k \times \log m)$	[HLSB06]
Insertion with Relaxation Variables	$\mathcal{O}(m)$	[MSL11]
Deletion with Model Rotation	$\mathcal{O}(m)$	[BLMS12,MSL11]
Progression	$\mathcal{O}(k \times \log(1 + \frac{m}{k}))$	[MSJB13]

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- Additional Techniques:
  - Restrict formula to unsatisfiable subsets [BDTW93,HLSB06,DHN06,MSL11]
  - Check redundancy condition [vMW08,MSL11,BLMS12]
  - Model rotation, recursive model rotation, etc. [MSL11,BMS11,BLMS12,W12]

Outline

CNF Encodings

SAT Embeddings

SAT Oracles

- MUS Extraction
- MaxSAT
- 2QBF

What Next in SAT-Based Problem Solving?

Defining Maximum Satisfiability

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable

Defining Maximum Satisfiability

$\boxed{x_6 \vee x_2}$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
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- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula

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- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula
- The MaxSAT solution is one of the smallest MCSes

MaxSAT Problem(s)

- MaxSAT:
  - All clauses are soft
  - Maximize number of satisfied soft clauses
  - Minimize number of unsatisfied soft clauses

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- Weighted MaxSAT
  - Weights associated with (soft) clauses
  - Minimize sum of weights of unsatisfied clauses

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Definitions

- Cost of assignment:
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- Optimum solution (OPT):
  - Assignment with minimum cost
- Upper Bound (UB):
  - Assignment with cost not less than OPT
  - E.g. $\sum_{c_i \in \varphi} w_i + 1$; hard clauses may be inconsistent
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  - E.g. -1; it may be possible to satisfy all soft clauses

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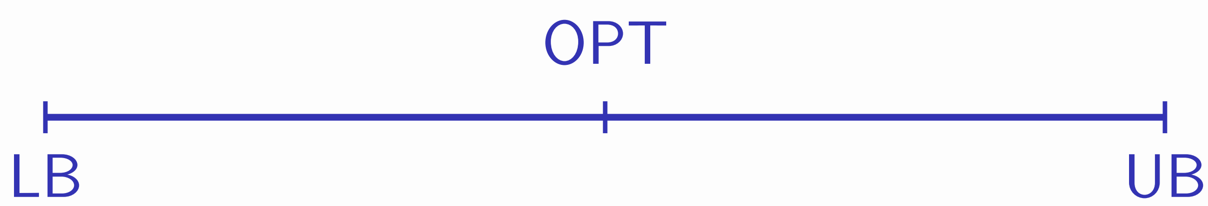
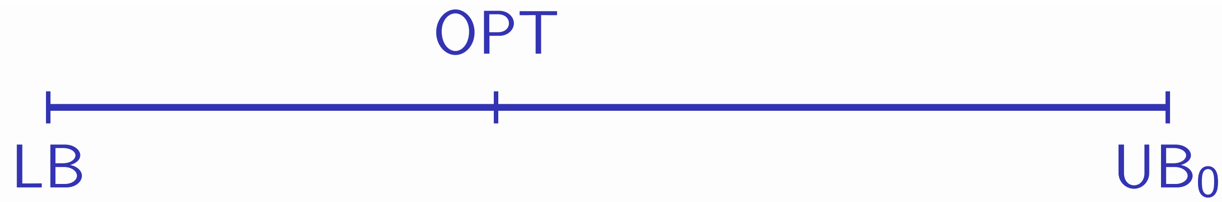


Iterative SAT Solving – Refine UB



- Require $\sum w_i \, r_i \leq UB_0 - 1$

Iterative SAT Solving – Refine UB

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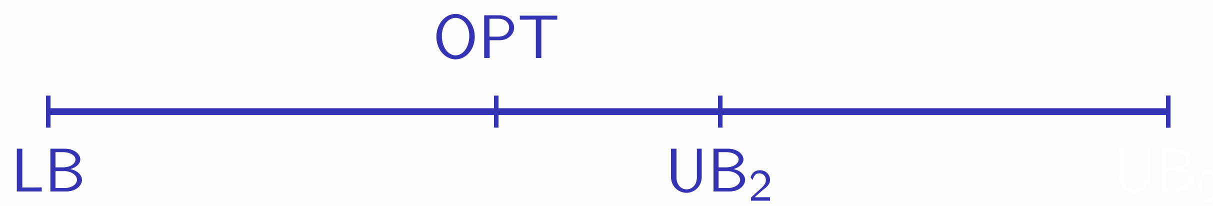

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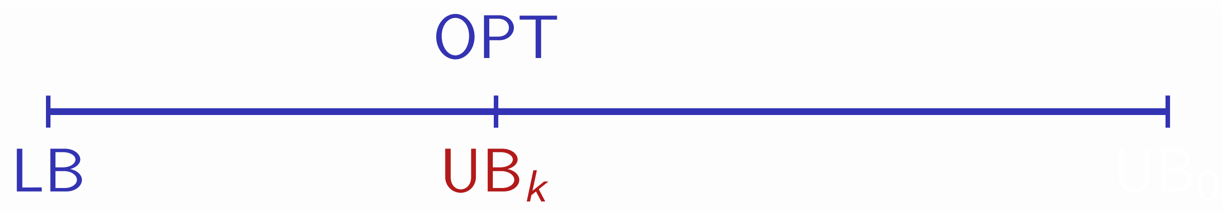

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- Repeat until constraint $\sum w_i\, r_i \leq UB_k - 1$ becomes UNSAT
  - UB_k denotes the optimum value

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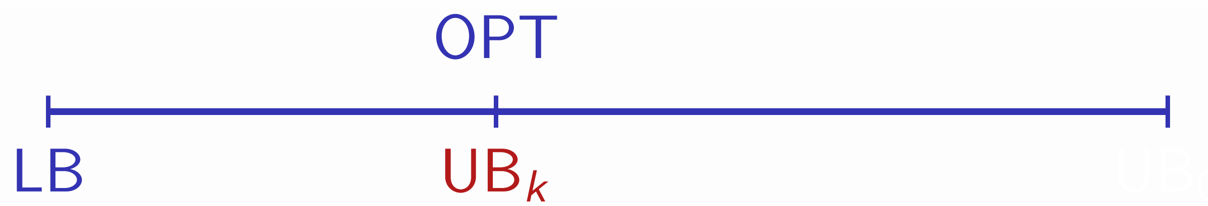

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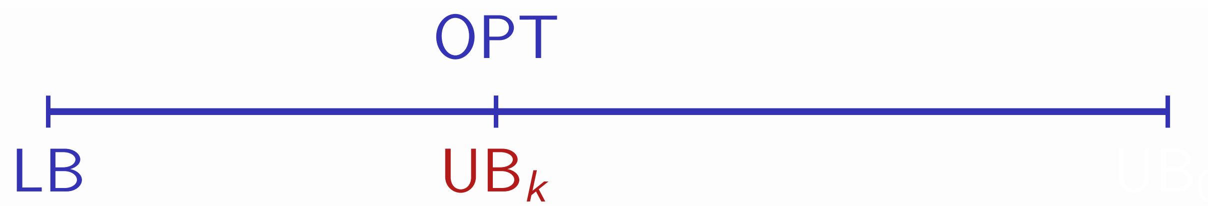

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- Worst-case # of iterations **exponential** on instance size

- Example tools:
  - Minisat+: CNF encoding of constraints [ES06]
  - SAT4J: native handling of constraints [LBP10]
  - QMaxSat: CNF encoding of constraints [KZFH12]
  - ...

Fu&Malik's Core-Guided Approach

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
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Example CNF formula

Fu&Malik's Core-Guided Approach

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Formula is UNSAT; $\mathrm{OPT} \leq |\varphi| - 1$; Get unsat core

Fu&Malik's Core-Guided Approach

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
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$\sum_{i=1}^{6} r_i \leq 1$

Add relaxation variables and AtMost1 constraint

Fu&Malik's Core-Guided Approach

$x_6 \vee x_2$ $\quad$ $\neg x_6 \vee x_2$ $\quad$ $\neg x_2 \vee x_1 \vee r_1$ $\quad$ $\neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad$ $x_6 \vee \neg x_8$ $\quad$ $x_2 \vee x_4 \vee r_3$ $\quad$ $\neg x_4 \vee x_5 \vee r_4$

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$\sum_{i=1}^{6} r_i \leq 1$

Formula is (again) UNSAT; OPT $\leq |\varphi| - 2$; Get unsat core

Fu&Malik's Core-Guided Approach

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1 \vee r_9$ $\quad \neg x_1 \vee r_2 \vee r_{10}$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_{11}$ $\quad \neg x_7 \vee x_5 \vee r_{12}$ $\quad \neg x_5 \vee x_3 \vee r_5 \vee r_{13}$ $\quad \neg x_3 \vee r_6 \vee r_{14}$

$\sum_{i=1}^{6} r_i \leq 1$ $\quad \sum_{i=7}^{14} r_i \leq 1$

Add new relaxation variables and AtMost1 constraint

Fu&Malik's Core-Guided Approach

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1 \vee r_9$	$\neg x_1 \vee r_2 \vee r_{10}$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
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Instance is now SAT

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$\sum_{i=1}^{6} r_i \leq 1$ $\quad$ $\sum_{i=7}^{14} r_i \leq 1$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

Organization of Fu&Malik's Algorithm

- Clauses characterized as:
  - Soft: initial set of soft clauses
  - Hard: initially hard, or added during execution of algorithm
    - E.g. clauses from AtMost1 constraints
- While exist unsatisfiable cores [FM06]
  - Add fresh set B of relaxation variables to soft clauses in core
  - Add new AtMost1 constraint

$$\sum_{b_i \in B} b_i \leq 1$$

    - At most 1 relaxation variable from set B can take value 1
- (Partial) MaxSAT solution is $|\varphi| - \mathcal{I}$
  - $\mathcal{I}$: number of iterations ($\equiv$ number of computed unsat cores)

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- (Partial) MaxSAT solution is $|\varphi| - \mathcal{I}$
  - $\mathcal{I}$: number of iterations ($\equiv$ number of computed unsat cores)
- Can be adapted for weighted MaxSAT [ABL09a,MMSP09]

Oracle-Based MaxSAT Solving I

- Iterative: [MHLPMS13]
  - Linear search SAT/UNSAT (refine UB) [e.g. LBP10]
  - Linear search UNSAT/SAT (refine LB)
  - Binary search [e.g. FM06]
  - Bit-based
  - Mixed linear/binary search [e.g. KZFH12]
- Core-Guided: [MHLPMS13,ABL13]
  - FM/(W)MSU1.X/WPM1 [FM06,MSM08,MMSP09,ABL09a,ABGL12]
  - (W)MSU3 [MSP07]
  - (W)MSU4 [MSP08]
  - (W)PM2 [ABL09a,ABL09b,ABL10,ABGL13]
  - Core-guided binary search (w/ disjoint cores) [HMMS11,MHMS12]
    - Bin-Core, Bin-Core-Dis, Bin-Core-Dis2
- Iterative subsetting [DB11,DB13a,DB13b]

Oracle MaxSAT Solving II

- A sample of recent algorithms:

Algorithm	# Oracle Calls	Reference
Linear search SU	Exponential	[e.g. LP10]
Binary search	Linear	[e.g. FM06]
WMSU1/WPM1	Exponential*	[FM06,MSM08,MMSP09,ABL09a,ABGL12]
WPM2	Exponential*	[ABL10,ABGL13]
Bin-Core-Dis	Linear	[HMMS11,MHMS12]
Iterative subsetting	Exponential	[DB11,DB13a,DB13b]

  * Weighted case; depends on computed cores

- Example MaxSAT solvers:
  - MSUnCore; WPM1, WPM2; QMaxSAT; SAT4J; etc.

Outline

Problem Statement

[GMN09]

Given: $\exists X \forall Y.\, \phi$, where ϕ is a propositional formula

Question: Is there an assignment τ to X such that $\forall Y.\, \phi[X/\tau]$?

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Example

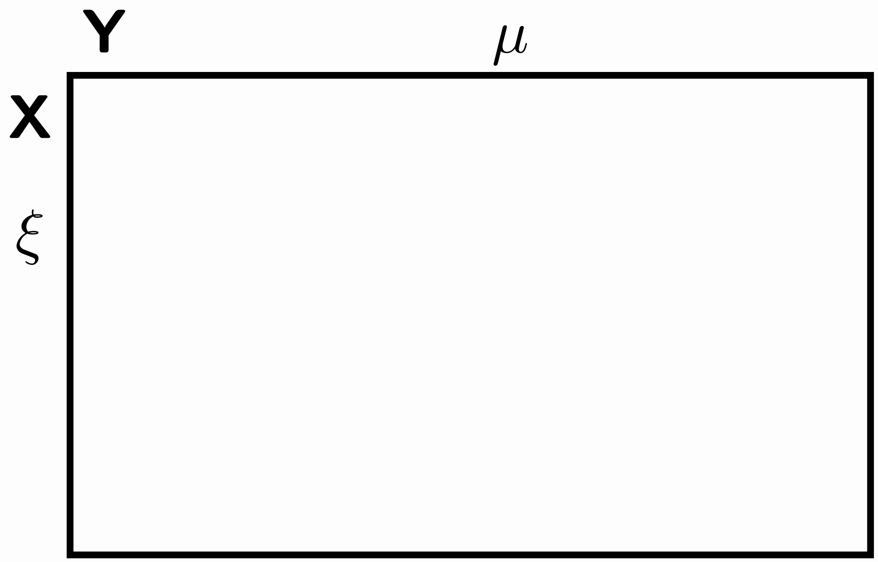
$$\exists x_1, x_2\ \forall y_1, y_2.\, (x_1 \rightarrow y_1) \wedge (x_2 \rightarrow y_2)$$

solution: $x_1 = 0, x_2 = 0$

Motivation

- Σ_2^P complete
- interesting problems in this class, e.g. certain nonmonotonic reasoning, aspects of model checking, conformant planning
- separate track at QBF Eval

Looking at Assignments



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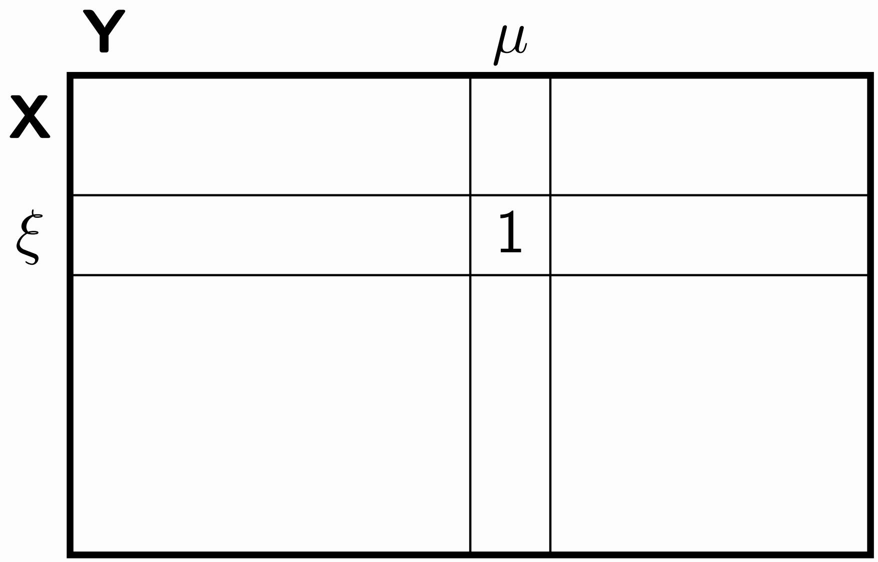

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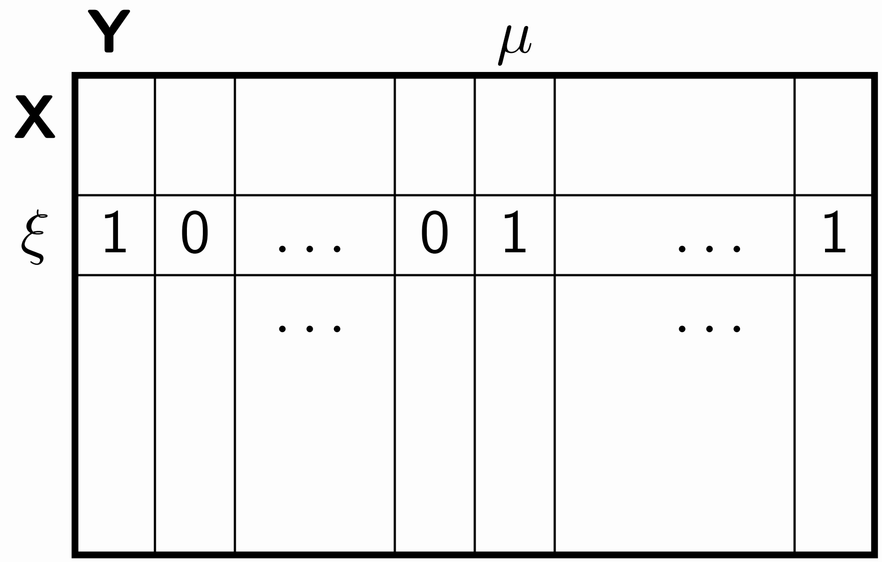

Looking at Assignments

	Y				μ		
X							
ξ	1	0	$\dots$	0	1	$\dots$	1
			$\dots$			$\dots$	
τ	1	1	$\dots$	1	1	$\dots$	1

Looking at Assignments

Y μ

X							
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			$\dots$			$\dots$	
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$\phi[Y/\mu]$

Expanding $\exists X \forall Y.\, \phi$ into SAT

$$\exists X \forall Y.\, \phi \;\longrightarrow\; \mathsf{SAT}\left(\bigwedge_{\mu \in \mathcal{B}^{|Y|}} \phi[Y/\mu] \right)$$

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$$\exists x_1, x_2 \forall y_1, y_2 .\, (x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$$

Expansion:

$$\begin{array}{ll} & (x_1 \leftrightarrow 0) \wedge (x_2 \leftrightarrow 0) \wedge (\bar{x}_1 \vee \bar{x}_2) \\ \wedge & (x_1 \leftrightarrow 0) \wedge (x_2 \leftrightarrow 1) \wedge (\bar{x}_1 \vee \bar{x}_2) \\ \wedge & (x_1 \leftrightarrow 1) \wedge (x_2 \leftrightarrow 0) \wedge (\bar{x}_1 \vee \bar{x}_2) \\ \wedge & (x_1 \leftrightarrow 1) \wedge (x_2 \leftrightarrow 1) \wedge (\bar{x}_1 \vee \bar{x}_2) \end{array}$$

Expanding $\exists X \forall Y .\, \phi$ into SAT

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Expansion:

$$\begin{array}{rl} & (x_1 \leftrightarrow 0) \wedge (x_2 \leftrightarrow 0) \wedge (\bar{x}_1 \vee \bar{x}_2) \\ \wedge & (x_1 \leftrightarrow 0) \wedge (x_2 \leftrightarrow 1) \wedge (\bar{x}_1 \vee \bar{x}_2) \\ \wedge & (x_1 \leftrightarrow 1) \wedge (x_2 \leftrightarrow 0) \wedge (\bar{x}_1 \vee \bar{x}_2) \\ \wedge & \color{red}{(x_1 \leftrightarrow 1) \wedge (x_2 \leftrightarrow 1) \wedge (\bar{x}_1 \vee \bar{x}_2)} \end{array}$$

Abstraction of $\exists X \forall Y.\, \phi$

- Consider only some set of assignments $\omega \subseteq \mathcal{B}^{|Y|}$

$$\bigwedge_{\mu \in \omega} \phi[Y/\mu]$$

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$$\bigwedge_{\mu \in \mathcal{B}^{|Y|}} \phi[Y/\mu] \;\Rightarrow\; \bigwedge_{\mu \in \omega} \phi[Y/\mu]$$

Abstraction of $\exists X \forall Y .\, \phi$

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- If a solution to the problem is a solution to the abstraction

$$\bigwedge_{\mu \in \mathcal{B}^{|Y|}} \phi[Y/\mu] \Rightarrow \bigwedge_{\mu \in \omega} \phi[Y/\mu]$$

- But not the other way around, a solution to an abstraction is not necessarily a solution to the original problem.

CEGAR Loop

input : $\exists X \forall Y . \phi$
output: (true, τ) if there exists τ s.t. $\forall Y . \phi[X/\tau]$,
$(\text{false}, -)$ otherwise

```
ω ← ∅;
while true do
    (outc₁, τ) ← SAT(⋀_{μ∈ω} φ[Y/μ]);          // find a candidate
    if outc₁ = false then
        return (false,–);                      // no candidate found
    end
    if "τ is a solution";                      // solution check
    then
        return (true, τ)
    else
        "Grow ω";                              // refinement
    end
end
```

CEGAR Loop

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        "Grow ω";                              // refinement
    end
end
```

Testing for Solution

A value τ is a solution to $\exists X \forall Y.\, \phi$ iff

$$\forall Y.\, \phi[X/\tau] \quad \textit{iff} \quad \mathsf{UNSAT}(\neg\phi[X/\tau])$$

Testing for Solution

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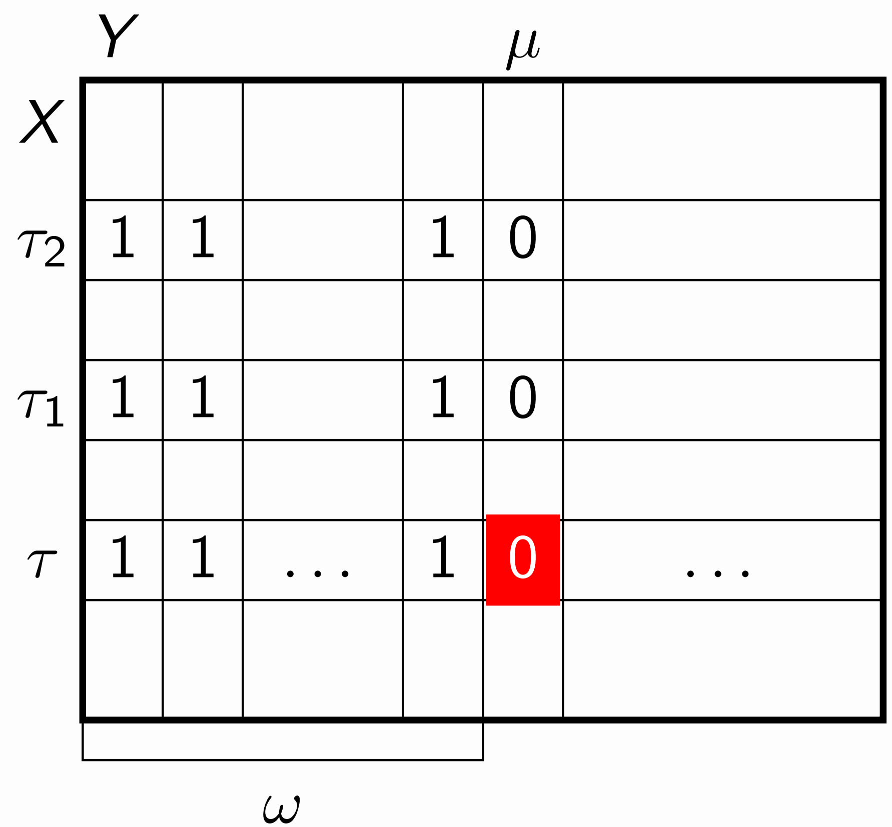
Example

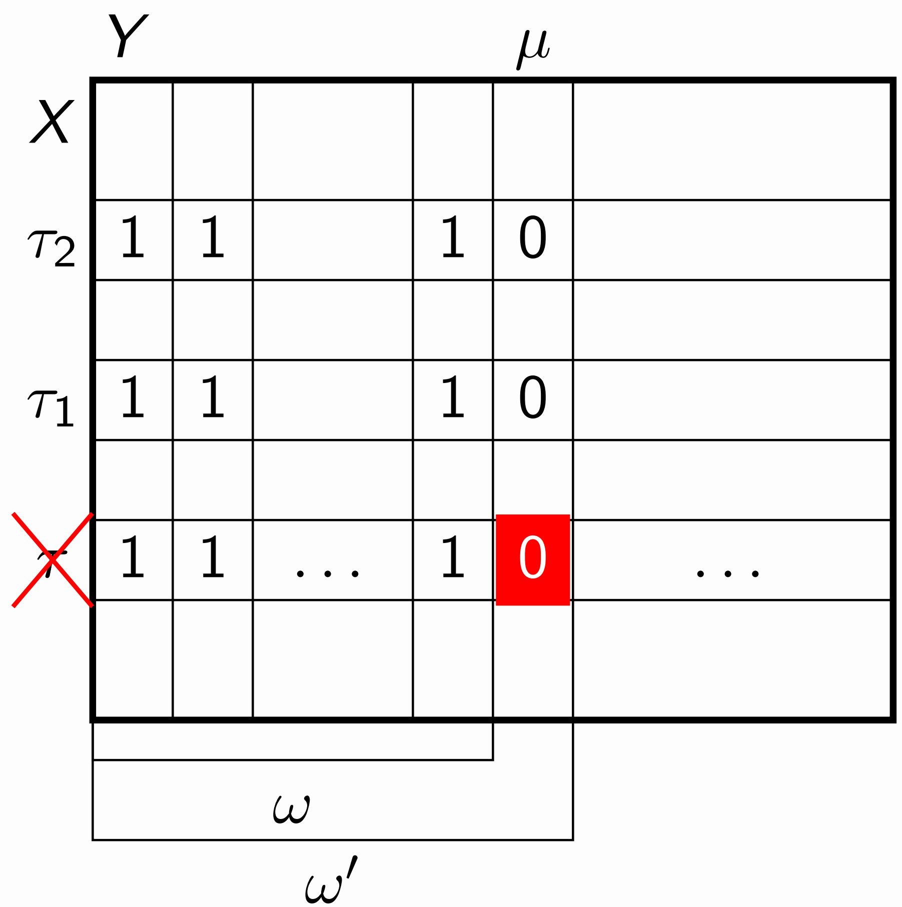
$\exists x_1, x_2\ \forall y_1, y_2.\ (x_1 \rightarrow y_1) \wedge (x_2 \rightarrow y_2)$

- candidate: $x_1 = 1, x_2 = 1$
- counterexamples: $y_1 = 0, y_2 = 0$
  $y_1 = 0, y_2 = 1$
  $y_1 = 1, y_2 = 0$

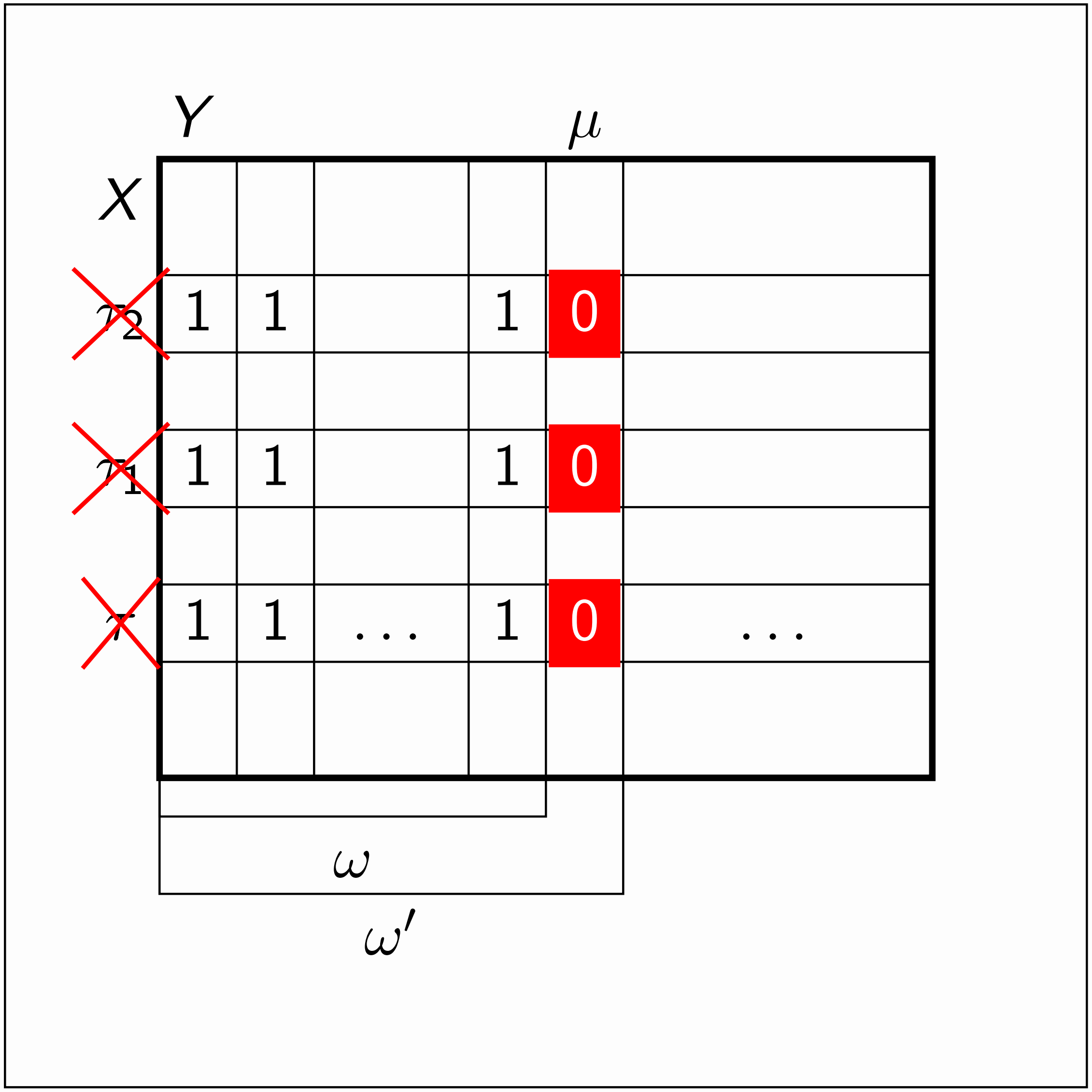
Refinement



Refinement



Refinement



AReQS (Abstraction Refinement-based QBF Solver)

input : $\exists X \forall Y . \phi$
output: (true, τ) if there exists τ s.t. $\forall Y . \phi[X/\tau]$,
$(\mathsf{false}, -)$ otherwise

```
ω ← ∅;                                  // start with the empty expansion
while true do
    (outc₁, τ) ← SAT(⋀_{μ∈ω} φ[Y/μ]);    // find a candidate
    if outc₁ = false then
        return (false,–);               // no candidate found
    end
    (outc₂, μ) ← SAT(¬φ[X/τ]);          // find a counterexample
    if outc₂ = false then
        return (true, τ) ;              // candidate is a solution
    end
    ω ← ω ∪ {μ} ;                       // refine
end
```

AReQS — Conclusions

- … is a CEGAR-based algorithm for 2QBF [JMS11]

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AReQS — Conclusions

- ... is a CEGAR-based algorithm for 2QBF [JMS11]
- ... uses SAT solver as an oracle
- ... gradually expands given 2QBF into a SAT formula
- Can be extended to arbitrary number of levels by recursion (RAReQS) [JKMSC12]

Outline

SAT-Based Problem Solving – A Glimpse of the Future

- Remarkable (and increasing) number of applications of SAT
- Can use SAT for solving problems in different complexity classes
  - $\text{FP}^{\text{NP}}[\log n]$, FP^{NP}, ...
  - E.g. tackling problems in the polynomial hierarchy
- Many new recent algorithms for concrete problems
  - MaxSAT
  - MUSes
  - MCSes
  - Enumeration problems
  - ...
- Better encodings?
- White-box vs. black-box approaches?
  - But use of oracles inevitable in many cases

Thank You

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