CIS 4930/6930: Principles of Cyber-Physical Systems

Chapter 2: Continuous Dynamics

Hao Zheng

Department of Computer Science and Engineering
University of South Florida

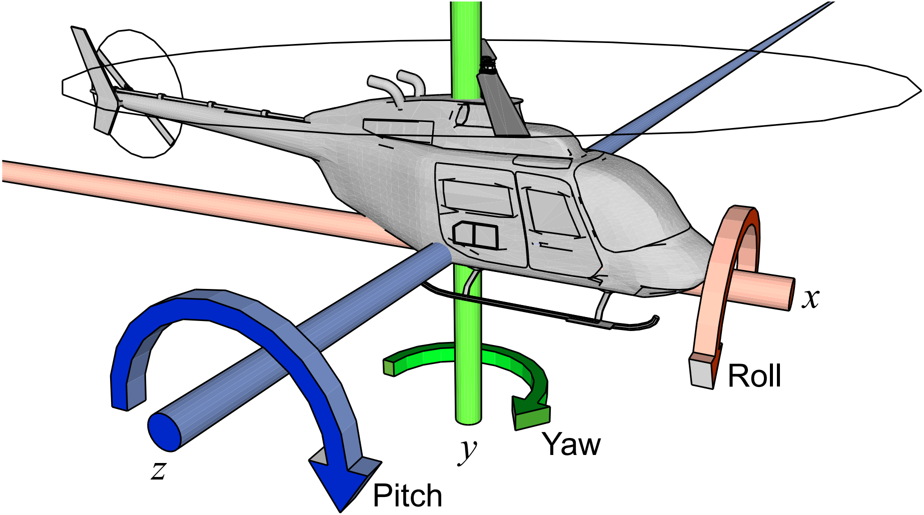
Modeling Techniques

- Models are abstractions of system dynamics (i.e., how things change over time):
- Examples:
  - Continuous dynamics - **ordinary differential equations** (ODEs)
  - Discrete dynamics - **finite-state machines** (FSMs)
  - **Hybrid systems** - a variety of hybrid system models

Modeling Continuous Dynamics

- **Classical mechanics** is the study of mechanical parts that move.
- Motion of mechanical parts can often be modeled using **ordinary differential equations** (ODEs).
- ODEs can also be applied to numerous other domains including circuits, chemical processes, and biological processes.
- ODEs used in tools such as `LabVIEW` (from National Instruments) and `Simulink` (from The MathWorks, Inc.).
- ODEs only work for "smooth" motion where *linearity*, *time invariance*, and *continuity* properties hold.
- Non-smooth motion, such as collisions, require **hybrid** (mixture of continuous and discrete) models (see next lecture).
- **Feedback control** can stabilize unstable systems.

2.1 Model of Helicopter Dynamics



Position

- Position is represented by six functions:

$$\begin{aligned} x &: \mathbb{R} \to \mathbb{R} \\ y &: \mathbb{R} \to \mathbb{R} \\ z &: \mathbb{R} \to \mathbb{R} \\ roll\ \theta_x &: \mathbb{R} \to \mathbb{R} \\ yaw\ \theta_y &: \mathbb{R} \to \mathbb{R} \\ pitch\ \theta_z &: \mathbb{R} \to \mathbb{R} \end{aligned}$$

  where the domain represents time and the co-domain (range) represents position or orientation along the axis.

- Collecting into two vectors:

$$\begin{aligned} \mathbf{x} &: \mathbb{R} \to \mathbb{R}^3 \\ \theta &: \mathbb{R} \to \mathbb{R}^3 \end{aligned}$$

  where **x** represents position and θ represents orientation.

Newton's Second Law

$$\mathbf{F}(t) = M\ddot{\mathbf{x}}(t)$$

where **F** is the force vector, *M* is the mass, and $\ddot{\mathbf{x}}$ is second derivative of **x** (i.e., the *acceleration*).

- **Velocity** can be determined as follows:

$$\begin{aligned} \forall t > 0, \quad \dot{\mathbf{x}}(t) &= \dot{\mathbf{x}}(0) + \int_0^t \ddot{\mathbf{x}}(\tau) d\tau \\ &= \dot{\mathbf{x}}(0) + \frac{1}{M}\int_0^t \mathbf{F}(\tau) d\tau \end{aligned}$$

- **Position** can be determined as follows:

$$\begin{aligned} \forall t > 0, \quad \mathbf{x}(t) &= \mathbf{x}(0) + \int_0^t \dot{\mathbf{x}}(\tau) d\tau \\ &= \mathbf{x}(0) + t\dot{\mathbf{x}}(0) + \frac{1}{M}\int_0^t \int_0^\tau \mathbf{F}(\alpha) d\alpha d\tau \end{aligned}$$

Rotational Version of Newton's Second Law

- The rotational version of force is **torque**:

$$\mathbf{T}(t) = \frac{d}{dt}(\mathbf{I}(t)\dot{\theta}(t))$$

$$\begin{bmatrix} T_x(t) \\ T_y(t) \\ T_z(t) \end{bmatrix} = \frac{d}{dt}\left(\begin{bmatrix} I_{xx}(t) & I_{xy}(t) & I_{xz}(t) \\ I_{yx}(t) & I_{yy}(t) & I_{yz}(t) \\ I_{zx}(t) & I_{zy}(t) & I_{zz}(t) \end{bmatrix} \begin{bmatrix} \dot{\theta}_x(t) \\ \dot{\theta}_y(t) \\ \dot{\theta}_z(t) \end{bmatrix} \right)$$

  where **T** is the torque vector and $\mathbf{I}(t)$ is the **moment of inertia tensor** that represents reluctance of an object to spin.

- When $\mathbf{I}(t)$ is a constant I, this reduces to:

$$\mathbf{T}(t) = I\ddot{\theta}(t)$$

Rotational Version of Newton's Second Law (cont)

- *Rotational acceleration*:

$$\ddot{\theta}(t) = \frac{\mathbf{T}(t)}{I}$$

- *Rotational velocity*:

$$\dot{\theta}(t) = \dot{\theta}(0) + \frac{1}{I}\int_0^t \mathbf{T}(\tau)d\tau$$

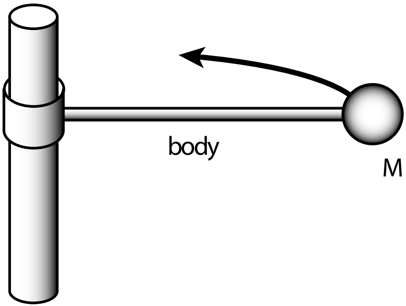
- Orientation:

$$\begin{aligned}\theta(t) &= \theta(0) + \int_0^t \dot{\theta}(\tau)d\tau \\ &= \theta(0) + t\dot{\theta}(0) + \frac{1}{I}\int_0^t \int_0^\tau \mathbf{T}(\alpha)d\alpha d\tau\end{aligned}$$

Feedback Control Problem

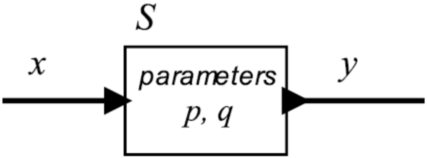
- A helicopter without a tail rotor will spin uncontrollably due to the torque induced by friction in the rotor shaft.
- Control system problem: apply torque using the tail rotor to counter the torque of the main rotor.

Model-Order Reduction: Simplified Helicopter Model



$$\ddot{\theta}_y(t) = T_y(t)/I_{yy}$$
$$\dot{\theta}_y(t) = \theta_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

2.2 Actor Model of Systems



- A *system* is a function that relates an input x to an output y:

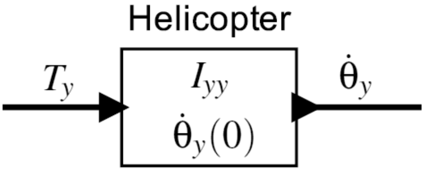
$$x : \mathbb{R} \to \mathbb{R}, \quad y : \mathbb{R} \to \mathbb{R}$$

- The domain and range of the system function are sets of signals, which are functions:

$$S : X \to Y$$

  where $X = Y = (\mathbb{R} \to \mathbb{R})$.
- Parameters may affect the definition of the function S.

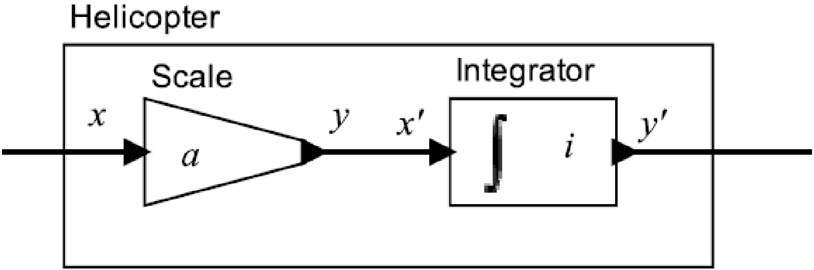
Actor Model of the Helicopter



- Input is the net torque of the tail rotor.
- Output is the angular velocity around the y axis.
- Parameters are I_{yy} and $\dot{\theta}_y(0)$.
- The system function is:

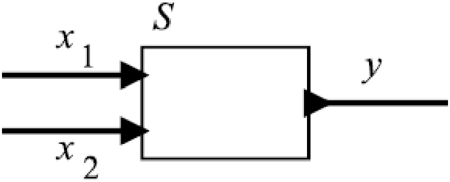
$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

Composition of Actor Models

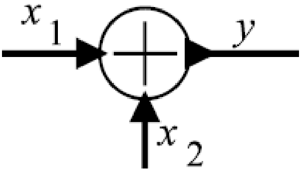


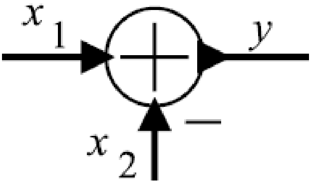
$$
\begin{aligned}
\forall t \in \mathbb{R}, \quad y(t) &= ax(t) \\
y &= ax \\
a &= 1/I_{yy} \\
\forall t \in \mathbb{R}, \quad y'(t) &= i + \int_0^t x'(\tau)d\tau \\
i &= \dot{\theta}_y(0)
\end{aligned}
$$

Actor Models with Multiple Inputs



$$S : (\mathbb{R} \to \mathbb{R})^2 \to (\mathbb{R} \to \mathbb{R})$$





$$\forall t \in \mathbb{R}, \quad y(t) = x_1(t) + x_2(t) \qquad y(t) = x_1(t) - x_2(t)$$

2.3 Properties of Systems

- Causal systems
- Memoryless systems
- Linearity and time invariance
- Stability

Causal Systems

- A system is **causal** if its output depends only on current and past inputs.
- Formally, a system is causal if for all $x_1, x_2 \in X$ and $\tau \in \mathbb{R}$:

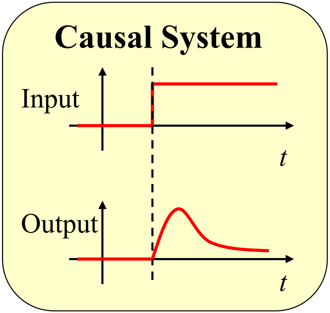
$$x_1|_{t \leq \tau} = x_2|_{t \leq \tau} \quad \Rightarrow \quad S(x_1)|_{t \leq \tau} = S(x_2)|_{t \leq \tau}$$

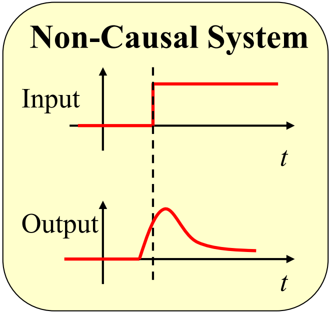
  where $x|_{t \leq \tau}$ is the **restriction in time** to current and past inputs.
- A system is causal if for two inputs x_1 and x_2 that are identical up to (and including) time τ, the outputs are identical up to (and including) time τ.
- A system is strictly causal if for all $x_1, x_2 \in X$ and $\tau \in \mathbb{R}$:

$$x_1|_{t < \tau} = x_2|_{t < \tau} \quad \Rightarrow \quad S(x_1)|_{t \leq \tau} = S(x_2)|_{t \leq \tau}$$

- $y(t) = x(t-1)$ is strictly causal, $y(t) = cx(t)$ is causal.
- Strictly causal actors are useful for constructing feedback systems.

Causal Systems





Most systems in nature are causal

Memoryless Systems

- A system has memory if the output depends not only on the current inputs, but also on past inputs (or future inputs, if not causal).
- In a **memoryless** system, the output at time t depends only on the input at time t .
- Formally, a system is memoryless if there exists a function $f : A \to B$ such that for all $x \in X$ and for all $t \in \mathbb{R}$:

$$(S(X))(t) \quad = \quad f(x(t))$$

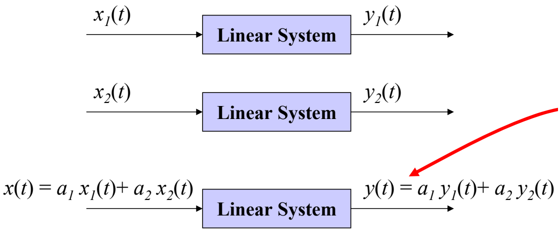
- The Integrator is not memoryless, but the adder is.
- A strictly causal, memoryless system has a constant output for all inputs.

Linearity and Time Invariance

- A system is **linear** if it satisfies the **superposition** property:

$$\forall x_1, x_2 \in X \text{ and } \forall a, b \in \mathbb{R}, S(ax_1 + bx_2) = aS(x_1) + bS(x_2)$$

- The helicopter example is linear if and only if $\dot{\theta}_y(0) = 0$.
- Integrator is linear when $i = 0$, and scale factor/adder are always linear.



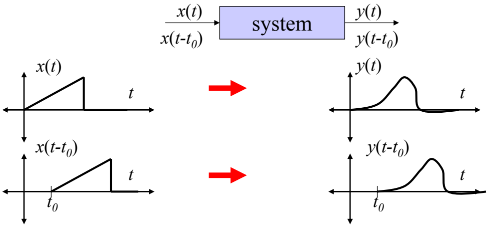
Linearity and Time Invariance

- A system is **time invariant** if:

$$\forall x \in X \text{ and } \tau \in \mathbb{R}, S(D_\tau(x)) = D_\tau(S(x))$$

where $D_\tau : X \to Y$ is a *delay* such that $(D_\tau(x))(t) = x(t - \tau)$.

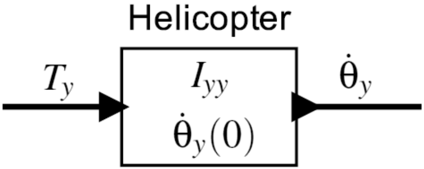
- Helicopter example is not time invariant unless no initial angular rotation, and the integral starts at $-\infty$.



Stability

- A system is **bounded-input bounded-output** (BIBO) *stable* if the output signal is bounded for all input signals that are bounded.
- Consider a continuous-time system with input w and output v.
- The input is bounded if there is a real number $A < \infty$ such that $|w(t)| \leq A$ for all $t \in \mathbb{R}$.
- The output is bounded if there is a real number $B < \infty$ such that $|v(t)| \leq B$ for all $t \in \mathbb{R}$.
- The system is stable if for any input bounded by some A, there is some bound B on the output.

Open-Loop Helicopter



- Helicopter example is not stable.
- Consider input $T_y = u$ where u is a *unit step* input:

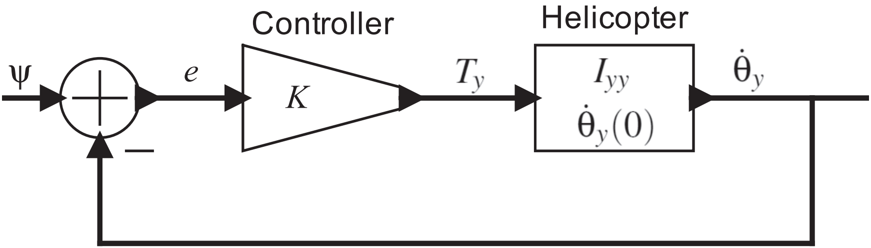
$$\forall t \in \mathbb{R}, \quad u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

- The system function is:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

2.4 Feedback Control

- **Feedback control** is used to achieve stability.
- These systems measure the error (difference between actual and desired behavior) and use this information to correct the behavior.



Mathematical Analysis

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$
$$= \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t (\Psi(\tau) - \dot{\theta}_y(\tau)) d\tau$$

Not easy to solve. Assume $\Psi = 0$.

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau$$

The solution:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) e^{-Kt/I_{yy}} u(t)$$

$\dot{\theta}_y(t)$ approaches 0 when t becomes large (K is positive).

Mathematical Analysis (cont)

Assume initially at rest with a non-zero desired angular velocity:

$$\begin{aligned} \dot{\theta}(0) &= 0 \\ \Psi(t) &= au(t) \end{aligned}$$

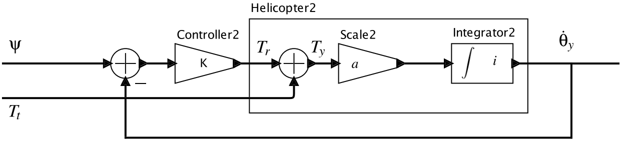
Substitute in and simplify as follows:

$$\begin{aligned} \dot{\theta}_y(t) &= \frac{K}{I_{yy}} \int_0^t (\Psi(\tau) - \dot{\theta}_y(\tau)) d\tau \\ &= \frac{K}{I_{yy}} \int_0^t a d\tau - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau \\ &= \frac{Kat}{I_{yy}} - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau \end{aligned}$$

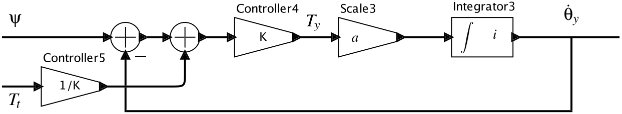
After some magic:

$$\dot{\theta}_y(t) = au(t)(1 - e^{-Kt/I_{yy}})$$

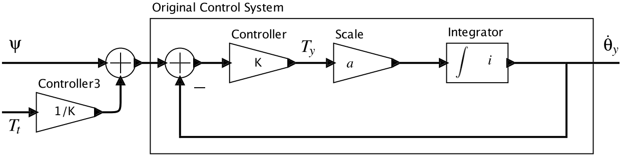
Helicopter Model with Separately Controlled Torques



(a)



(b)



(c)

Mathematical Analysis

Suppose the torque to the top rotor is:

$$T_t = bu(t)$$

Suppose the desired angular rotation is:

$$\Psi(t) = 0$$

Input to the original control system is:

$$x(t) = \Psi(t) + T_t(t)/K = (b/K)u(t)$$

The solution is:

$$\dot{\theta}_y(t) = (b/K)u(t)(1 - e^{-Kt/I_{yy}})$$

Concluding Remarks

- This lecture introduces two modeling techniques that describe physical dynamics: ODEs and actor models.
- This lecture emphasizes the relationship between these models.
- The fidelity of a model (how well it approximates the system) is a strong factor in the success or failure of any engineering effort.

Continuous vs Discrete Signals

- **Continuous**, also called **continuous time**, signals defined as

$$x : \mathbb{R} \to \mathbb{R}$$

- **Discrete**, also called **discrete time**, signals defined as

$$x : \mathbb{Z} \to \mathbb{Z}$$