Summary of Key Probability Distributions

This handout contains a summary of some important probability distributions. The distributions summarized here are uniform (continuous), uniform (discrete), binomial, Poisson, exponential, Pareto, and bounded Pareto.

<u>Uniform distribution (continuous):</u>

A random variable uniformly distributed in $a \le x \le b$ has probability density function,

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and a probability distribution function,

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{(x-a)}{(b-a)} & a \le x < b \\ 1 & x \ge b \end{cases}$$

The mean and variance are,

$$\mu = \frac{1}{2}(a+b) \text{ and } \sigma^2 = \frac{1}{12}(b-a)^2$$

To generate use `genunifc.c` from Christensen tools page or uniform() in CSIM.

<u>Uniform distribution (discrete):</u>

A random variable uniformly distributed in $a \le k \le b$ where $n = b - a + 1$ has a probability mass function,

$$f(k) = \begin{cases} 1/n & a \le k \le b \\ 0 & \text{otherwise} \end{cases}$$

and a cumulative distribution function,

$$F(k) = \begin{cases} 0 & k < a \\ (\lfloor k \rfloor - a + 1)/n & a \le k \le b \\ 1 & k > b \end{cases}$$

The mean and variance are,

$$\mu = \frac{1}{2}(a+b) \text{ and } \sigma^2 = \frac{1}{12}(n^2 - 1)$$

To generate use `genunifd.c` from Christensen tools page or random_int() in CSIM.

Binomial distribution (discrete):

A random variable binomially distributed for n trials with probability p of success for each trial has probability mass function,

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \le k \le n$$

The cumulative distribution function is messy.

The mean and variance are,

$$\mu = np \text{ and } \sigma^2 = np(1-p)$$

Note: Define $\lambda = np$, then as n goes to infinity the binomial distribution tends to the Poisson distribution with rate λ.

To generate use `genbin.c` from Christensen tools page or binomial() in CSIM.

Poisson distribution (discrete):

A random variable Poisson distributed for a rate λ of arrivals has probability mass function,

$$f(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k = 0, 1, 2, \ldots$$

The cumulative distribution function is messy.

The mean and variance are,

$$\mu = \lambda \text{ and } \sigma^2 = \lambda$$

Note: The distribution of time between arrivals in Poisson process is exponentially distributed with mean $1/\lambda$.

To generate use `genpois.c` from Christensen tools page or poisson() in CSIM.

Exponential distribution (continuous):

A random variable exponentially distributed has density function,

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t \le 0 \end{cases}$$

and distribution function,

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & x > 0 \\ 0 & x \le 0 \end{cases}$$

The mean and variance are,

$$\mu = \frac{1}{\lambda} \text{ and } \sigma^2 = \frac{1}{\lambda^2}$$

To generate use `genexp.c` from Christensen tools page or exponential() in CSIM.

Pareto distribution (continuous):

A random variable Pareto distributed with shape parameter α and minimum value k has density function,

$$f(x) = \begin{cases} \dfrac{\alpha \cdot k^{\alpha}}{x^{\alpha+1}} & x \geq k \\ 0 & x < k \end{cases}$$

and distribution function,

$$F(x) = \begin{cases} 1 - \left(\dfrac{k}{x}\right)^{\alpha} & x \geq k \\ 0 & x < k \end{cases}$$

The mean and variance are,

$$\mu = \frac{\alpha \cdot k}{\alpha - 1} \text{ and } \sigma^2 = \frac{\alpha \cdot k^2}{\left((\alpha-1)^2(\alpha-2)\right)}$$

Note: The Pareto distribution is heavy tailed. The mean is infinity for $\alpha < 1$ and the variance is infinity for $\alpha < 2$.

To generate use `genpar1.c` from Christensen tools page or pareto() in CSIM.

Bounded Pareto distribution (continuous):

A random variable X Bounded Pareto distributed with shape parameter α, minimum value k, and maximum value p has density function,

$$f(x) = \begin{cases} 0 & x < k \\ \dfrac{\alpha \cdot k^{\alpha}}{\left(1 - \left(\dfrac{k}{p}\right)^{\alpha}\right)} x^{-\alpha-1} & k \leq x \leq p \\ 0 & x > p \end{cases}$$

and distribution function,

$$F(x) = \begin{cases} 0 & x < k \\ \dfrac{p^{\alpha} \cdot \left(k^{\alpha} - x^{\alpha}\right)}{x^{\alpha} \cdot \left(k^{\alpha} - p^{\alpha}\right)} & k \leq x \leq p \\ 0 & x > p \end{cases}$$

The mean and variance are,

$$\mu = \frac{\alpha\left(k^{\alpha} \cdot p^{1-\alpha} - k\right)}{(\alpha-1)\left(\left(\dfrac{k}{p}\right)^{\alpha} - 1\right)} \text{ and } \sigma^2 = \frac{\alpha \cdot \left(k^{\alpha} \cdot p^{2-\alpha} - k^2\right)}{(\alpha-2) \cdot \left(\left(\dfrac{k}{p}\right)^{\alpha} - 1\right)} - \frac{\alpha^2 \cdot \left(k - k^{\alpha} \cdot p^{1-\alpha}\right)^2}{(\alpha-1)^2 \cdot \left(\left(\dfrac{k}{p}\right)^{\alpha} - 1\right)^2}$$

Note: The Bounded Pareto distribution is effectively heavy tailed, but has finite mean and variance.

To generate use `genpar2.c` from Christensen tools page (there is no built-in CSIM function to generate Bounded Pareto random variables).
